TECHNOLOGY DEPARTMENT

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 61 NUMBER 7

PUBLIC LIBRARY DETROIT

PART I

CONTENTS

Recent Applications of Convex Functions . . . J. W. Green 449
Inverse and Complementary Sequences of Natural Numbers . . . J. Lambek and L. Moser 454
Carnot and the Concept of Deviation . . . . . C. B. Boyer 459
Mathematical Notes . . . . . A. Oppenheim 463
Classroom Notes. . . . . W. T. Guy, Jr., L. I. Lowell 466
Elementary Problems and Solutions 470
Advanced Problems and Solutions 476
Recent Publications 483
News and Notices 494
The Mathematical Association of America 498
New Members. 499
March Meeting of the Metropolitan New York Section . . . 501
March Meeting of the Michigan Section 503
March Meeting of the Southeastern Section 505
April Meeting of the Iowa Section 513
April Meeting of the Ohio Section 516
April Meeting of the Rocky Mountain Section 519
April Meeting of the Southwestern Section 522

AUGUST–SEPTEMBER 1954

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

Carl B. Allendoerfer, *Editor*

ASSOCIATE EDITORS

R. V. CHURCHILL	EDWIN HEWITT	GEORGE SPRINGER
HOWARD EVES	L. M. KELLY	E. P. STARKE
F. A. FICKEN	IVAN NIVEN	G. B. THOMAS
WALLACE GIVENS	EDITH R. SCHNECKENBURGER	E. P. VANCE

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. Allendoerfer, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. Randolph, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

OFFICERS OF THE ASSOCIATION

President, E. J. McShane, University of Virginia
Honorary President, W. D. Cairns, Oberlin College
First Vice-President, H. S. M. Coxeter, University of Toronto
Second Vice-President, W. L. Duren, Jr., Tulane University
Secretary-Treasurer, H. M. Gehman, University of Buffalo
Associate Secretary, Edith R. Schneckenburger, University of Buffalo
Editor, C. B. Allendoerfer, University of Washington

Additional Members of the Board of Governors: I. A. Barnett, M. A. Basoco, B. H. Brown, S. S. Cairns, H. H. Downing, L. R. Ford, R. M. Foster, Philip Franklin, R. F. Graesser, E. R. Heineman, R. C. Huffer, Ralph Hull, C. A. Hutchinson, S. B. Jackson, R. D. James, L. W. Johnson, P. S. Jones, R. E. Langer, D. H. Lehmer, F. A. Lewis, Saunders MacLane, K. O. May, R. J. Michel, E. B. Miller, W. E. Milne, C. W. Munshower, W. H. Myers, C. O. Oakley, Morris Ostrofsky, G. B. Price, F. A. Rickey, C. E. Springer, C. W. Trigg, A. W. Tucker

---

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

RECENT APPLICATIONS OF CONVEX FUNCTIONS*

J. W. GREEN, University of California at Los Angeles

1. Introduction. A comprehensive account of the history, properties, and applications of convex functions up to 1946 has been given by E. F. Beckenbach [1]. However, even in the few years since that article appeared, convex functions have been so consistently of value in analysis, geometry, and other branches of mathematics, notably mathematical economics, that a considerable wealth of material exists for a supplementary article. In the following I will report on some of this material. Coverage of the field will not be exhaustive; the choice of subject matter was determined by my own fields of interest; namely, convex functions *per se*, and their generalizations, geometry of convex figures, function theory, and (through various logistics and programming projects) linear and non-linear programming.

2. Convex functions and generalizations. We recall that a convex function f is one which satisfies the inequality

$$f(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) \leqq \lambda f(\mathbf{x}) + (1 - \lambda) f(\mathbf{y})$$

for all λ such that $0 \leqq \lambda \leqq 1$. The variables $\mathbf{x}$ and $\mathbf{y}$ may be regarded as vectors in any linear vector space, which in this article will be taken to be Euclidean. This inequality states that between any two points of the graph of f, the graph lies nowhere above the chord joining those two points. For elementary properties of convex functions, see [1] and [2].

Now, as far as the study of properties of convex functions, regarded as functions of one or more real variables, is concerned, there has not been a great deal of activity of late. This is due perhaps in one dimension to a shortage of unsolved problems and in more dimensions to a shortage of solvable ones. Here we shall describe several generalizations of convex functions which seem interesting or useful.

First, let us stay in one dimension and suppose that there is a family $\mathcal{F}$ of continuous functions F such that through any two points (x_1, y_1) and (x_2, y_2) with $x_1 \neq x_2$ we may interpolate exactly one function F of $\mathcal{F}$. A given function f is said to be convex relative to $\mathcal{F}$, or sub-F, provided it agrees with F at x_1 and x_2, it lies nowhere above F for x between x_1 and x_2. It is clear that the functions F play the role which the linear functions play in the ordinary convexity. The sub-F functions were studied extensively by Beckenbach and Bing [3] and were reported on in [1]. However in the recent past, a number of papers [4, 5, 6] have appeared concerning these functions, particularly in connection with

* This article was originally prepared to be presented at the Christmas, 1952, meeting of the Association in St. Louis. However, an overly convex appendix on Christmas eve made its presentation in person impossible.

their smoothness and convergence properties. For example if the F are sufficiently well-behaved, the sub-F functions have very nearly the same differentiability properties as the convex functions—unilateral derivatives exist everywhere, derivatives everywhere with countable exceptions, second derivatives almost everywhere. A convergent sequence of sub-F functions converges uniformly, and the sequence of derivatives converges boundedly.

Now the sub-F functions, instead of being only generalizations for generalization's sake, have proved to be applicable to a number of problems. In fact, a particular case of them, the sub-trigonometric functions for which $\mathfrak{F}$ consists of the functions $A \cos x + B \sin x$, were introduced by Phragmén and Lindelöf and used by Pólya in connection with problems in function theory. Pólya [7] proved that a function $p(\theta)$ of period 2π is the support function of a convex area in the plane if and only if it is sub-trigonometric. The present author rediscovered this fact [8] and made use of it in several geometrical problems (see §3 below). In [8], proper acknowledgement is not made to Pólya, and the author wishes to make it here. In [6], by use of the differentiability and convergence properties mentioned above, a simple proof is given of the known but apparently not easily accessible fact that the area of the general convex curve is given by $\frac{1}{2}\int_0^{2\pi} (p^2 - p'^2)d\theta$ without any smoothness assumptions.

Another and novel generalization of convex functions has been given by Hyers and Ulam [9]. They call a function approximately convex, or ϵ-convex provided it satisfies the inequality

$$f(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) \leqq \lambda f(\mathbf{x}) + (1 - \lambda) f(\mathbf{y}) + \epsilon,$$

where $\epsilon > 0$ and $\mathbf{x}$, $\mathbf{y}$ are vectors in n-space. The chief theorem about an ϵ-convex function is that it is approximately equal to a convex function. More precisely, there is a convex function g such that $g \leqq f \leqq g + k_n\epsilon$, where k_n depends only on the dimension n. The best value for k_n has not been determined for all n. For $n = 1$, it is 1, and it is easily seen that in one dimension, the most general ϵ-convex function is obtained by adding to an arbitrary convex function an arbitrary function whose values lie between 0 and ϵ. In [9], the value $k_n = (n^2 + 3n)/(2n + 2)$ is given, which is the best value for $n = 1, 2$, but not beyond. The present author [10] has obtained an improved value of k_n asymptotic to $\log_2 n$; this value is the best possible through $n = 3$ but beyond that the question is open.

A third type of function akin to convex and studied of late is the sub-additive function, satisfying the inequality $f(x+y) \leqq f(x) + f(y)$. Rosenbaum [11] has studied these functions systematically. If f is sub-additive and homogeneous, it is convex; if f is convex for $x \geqq 0$ and $f(0) \geqq 0$, it is sub-additive. These functions have many properties analogous to those of convex functions, but behave quite differently as far as continuity properties are concerned. For example, there exist measurable everywhere discontinuous sub-additive functions.

3. Geometry. Each year dozens of papers on the geometry of convex bodies are written, and these will usually involve convex functions, directly or indirectly. However here are a few problems where the analysis of functions is used to produce geometric results.

In [8] the following problem was considered: Let K be a convex curve lying interior to a circle C. What can be said about K if it subtends at each point of C the same angle α? It was shown that if $\pi-\alpha$ is an irrational multiple of π or is of the form $(m/n)\pi$ with m even and prime to n, then K must be a circle concentric to C; in other cases—for example $\alpha=\pi/2$—there is a considerable variety of curves K.* The author used the sub-trigonometric characterization of the supporting function $p(\theta)$ of K to solve several external problems involving perimeter, area, diameter, and width of K for those α where there is any problem. Since, for example, the length is given by $\int_0^{2\pi} p(\theta)d\theta$, it is not surprising that this was possible. A typical result is that the maximum length of K is attained by a figure consisting of n elliptical arcs pieced together with n corners of angle π/n.

The same characterization also proved useful in connection with another geometrical problem. Let K be a convex curve, and let the plane be subjected to an area-preserving affine transformation T so as to minimize the length of TK. In [12] the present author showed that K minimizes the length if and only if the Fourier coefficients a_2 and b_2 of $p(\theta)$ vanish. The problem was then posed of finding the curve of area 1 which when transformed so as to minimize the length gives the greatest length. The partial result that the extreme figure must be a polygon of five or fewer sides was obtained by using variations of $p(\theta)$ consistent with its sub-trigonometric nature and with the fact that $a_2=b_2=0$. The fact that it must be a triangle was subsequently proved by Gustin [13].

It is interesting that new proofs of geometric theorems have been obtained by methods originally devised in connection with mathematical economics. Karlin and Shapley [14] have employed the following theorem, due to Bohnenblust and themselves [15]: Let $\phi_\alpha(\mathbf{x})$ be a family of continuous convex functions in a compact convex set Δ in n-space with $\inf_{\mathbf{x}\in\Delta}\sup_\alpha\phi_\alpha(\mathbf{x})>0$. Then there exists a linear combination of $n+1$ of the ϕ_α with non-negative coefficients which is positive in Δ. Using this fact, Karlin and Shapley gave a new proof of Helly's theorem and also of the theorem that if a spherical surface in n-space is covered by a compact set of hemispherical sub-surfaces, then it is covered by a certain $n+1$ of these hemispheres.

4. Function theory. E. F. Beckenbach [16; references to related problems may be found therein] has recently obtained an interesting result relating the mean values of the modulus of a function analytic within a unit circle along radii and along concentric circles. Let

* It is curious that Santaló [20] has shown that if the very same problem is considered on the surface of a sphere instead of in the plane, one is led to the conclusion that K is in all cases a circle, no matter what α is.

$$\nu(\phi) = \int_0^1 |f(re^{i\phi})|\, dr, \qquad \mu(r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\phi})|\, d\phi.$$

Then if $\nu(\phi) \leqq 1$, (or even if the average of ν is $\leqq 1$) then $\mu(r) \leqq |f(0)| + 2r(1-|f(0)|)$ for $r \leqq \frac{1}{2}$ and $\mu(r) \leqq |f(0)| + (1-|f(0)|)/2(1-r)$ for $\frac{1}{2} \leqq r \leqq 1$. This result followed from the fact that $\mu(r)$ is a convex function and $\int_0^{2\pi} \mu(r)dr \leqq 1$. These two facts, without further reference to analytic function theory, imply the stated inequalities on μ. It seems quite remarkable that inequalities such as these can be achieved so directly by consideration of simple geometric properties of convex functions.

5. Programming. The typical problem of linear programming is that of minimizing a linear function $g(\mathbf{x}) = \sum_1^n c_i x_i$ of n variables $x_1, x_2, \cdots, x_n$, subject to a certain set of m linear constraints, $f_j(\mathbf{x}) \equiv b_j - \sum_{i=1}^n a_{ji}x_i \geqq 0$, $j = 1, 2, \cdots, m$ ($\mathbf{x}$ means the vector with components $x_1, \cdots, x_n$) and also the n special linear constraints $x_i \geqq 0$. A problem readily cast in this form is that of minimizing the cost g of a diet composed of n foodstuffs consumed in quantities $x_1, \cdots, x_n$ and containing m basic dietary ingredients. The inequalities $f_j \geqq 0$ correspond to stating minimum requirements of the various ingredients. The interpretation of $x_i \geqq 0$ is clear.

The above problem is that of determining the maximum of a linear (and thus convex) function in a convex polyhedron. In fact, it amounts to finding the distance from the origin to the supporting plane to this polyhedron normal to the vector $(c_1, c_2, \cdots, c_n)$. It may be stated in the form of a minimax or saddle value problem by the introduction of Lagrangian multipliers as follows [17]. First define $\phi(\mathbf{x}, \lambda) = g(\mathbf{x}) + \sum_1^m \lambda_j f_j(\mathbf{x})$. A vector $\mathbf{x}^0$ yields a solution of the maximum problem if and only if there exists a vector λ^0 with nonnegative components such that

$$\phi(\mathbf{x}, \lambda^0) \leqq \phi(\mathbf{x}^0, \lambda^0) \leqq \phi(\mathbf{x}^0, \lambda)$$

for $\lambda_j \geqq 0$, $x_i \geqq 0$; that is, if $(\mathbf{x}^0, \lambda^0)$ is a saddle point of the "surface" $z = \phi(\mathbf{x}, \lambda)$.

Related to the above problem is the dual problem; namely that of determining the minimum of $\sum_1^m b_j\lambda_j$ subject to the constraints $\lambda_j \geqq 0$, $\sum_1^m a_{ji}\lambda_j \geqq c_i$. One sees immediately that this problem leads to identically the same saddle problem as before, and indeed the two extremum problems are equivalent; if one has a solution, the other does, and the one minimum equals the other maximum.

Fenchel [18, 19], dropping linearity entirely, has made a very broad generalization of these notions, based upon a certain interesting conjugate relation between convex functions. This conjugate relation depends on the pole-polar relation with respect to the quadric

$$2z = x_1^2 + x_2^2 + \cdots + x_n^2 = \mathbf{x}\cdot\mathbf{x}$$

in $(n+1)$ space.

Let f be a convex function of $\mathbf{x}=(x_1, \cdots, x_n)$ in a convex set D. The polar plane of a point $(\mathbf{x}, z)$ is given by

$$z + \zeta = \mathbf{x}\cdot\xi,$$

where $(\xi, \zeta)=((\xi_1, \cdots, \xi_n), \zeta)$ are coordinates in another $(n+1)$ space in which it is convenient to imagine the polar plane. Then the function $\phi(\xi)$ conjugate to $f(\mathbf{x})$ is determined as the upper envelope of these polar planes as $(\mathbf{x}, z)$ varies over the graph $z=f(\mathbf{x})$; that is,

$$\phi(\xi) = \sup_{x\in D} (\mathbf{x}\cdot\xi - f(\mathbf{x})).$$

The domain Δ of definition of ϕ is those points where the sup is finite. Now Fenchel shows (under certain continuity conditions on f) that Δ is a convex set and ϕ a convex function; furthermore the conjugate of ϕ is f.

Now let f be convex in C and g concave in D, where C and D are convex sets. Let ϕ in Γ and ψ in Δ be the conjugate functions of f and g respectively ($-\psi$ is conjugate to $-g$). Then the following two extremal problems are equivalent: (a) maximize $g(x)-f(x)$ in $C\cap D$; (b) minimize $\phi(\xi)-\psi(\xi)$ in $\Gamma\cap\Delta$. Either both or neither of the problems have solutions, and the one maximum equals the other minimum. In case $g(\mathbf{x})=\sum c_i x_i$ in the positive orthant D: $x_i\geqq 0$, and $f(\mathbf{x})=0$ in the convex region C determined by the inequalities $\sum a_{ji}x_i\geqq b_j$, this pair of problems reduces to the dual linear programming problems mentioned earlier.

It would appear that this duality should be capable of further exploitation. Incidentally, Fenchel [19] shows how it can be used to prove many of the classical inequalities—Hölder's, *etc.*

References

1. E. F. Beckenbach, Convex functions, Bull. Amer. Math. Soc., vol. 54, 1948, pp. 439–460.
2. T. Bonnesen and W. Fenchel, Theorie der konvexen Körper, Berlin, Springer, 1934.
3. E. F. Beckenbach and R. H. Bing, On generalized convex functions, Trans. Amer. Math. Soc., vol. 58, 1945, pp. 220–230.
4. M. Matos-Peixoto, On the existence of derivatives of generalized convex functions, Summa Brasil Math., vol. 2, no. 3, 1948, pp. 35–42.
5. F. F. Bonsall, The characterization of generalized convex functions, Quart. J. Math., Oxford Ser. 2, vol. 1, 1950, pp. 100–111.
6. J. W. Green, Support, differentiability, and convergence properties of generalized convex functions. Proc. Amer. Math. Soc., vol. 4, 1953, pp. 391–396.
7. G. Pólya, Untersuchungen über Lücken und Singularitaten von Potenzreihen. Math. Zeit., vol. 29, 1929, pp. 549–560.
8. J. W. Green, Sets subtending a constant angle on a circle, Duke Math. J., vol. 17, 1950, pp. 263–267.
9. D. H. Hyers and S. N. Ulam, Approximately convex functions, Proc. Amer. Math. Soc., vol. 3, 1952, pp. 821–828.
10. J. W. Green, Approximately convex functions, Duke Math. J., vol. 19, 1952, pp. 499–504.
11. R. A. Rosenbaum, Sub-additive functions, Duke Math. J., vol. 17, 1950, pp. 227–247.
12. J. W. Green, Length and area of a convex curve under area-preserving affine transformation, Pac. J. Math., vol. 3, 1953, pp. 393–402.

13. W. S. Gustin, An isoperimetric minimax, Pac. J. Math., vol. 3, 1953, pp. 403–405.

14. S. Karlin and L. S. Shapley, Some applications of a theorem on convex functions, Ann. of Math., 2, vol. 52, 1950, pp. 148–153.

15. H. F. Bohnenblust, S. Karlin, and L. S. Shapley, Games with continuous, convex pay-off, Annals of Mathematics Studies 24, 1950, pp. 181–192. Princeton University Press.

16. E. F. Beckenbach, A property of mean values of an analytic function, Rend. Circ. Mat. Palermo II, vol. 1, 1952, pp. 1–7.

17. H. W. Kuhn and A. W. Tucker, Nonlinear programming, Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, pp. 481–492. University of California Press, 1951.

18. W. Fenchel, On conjugate convex functions, Canadian J. Math., vol. 1, 1949, pp. 73–77.

19. W. Fenchel, Convex Cones, Sets, and Functions, Logistics Project Report (multilith), Department of Mathematics, Princeton University, 1953.

20. L. A. Santaló, Two characteristic properties of circles on a spherical surface, Math. Notae, vol. 11, 1951, pp. 73–78.

INVERSE AND COMPLEMENTARY SEQUENCES OF NATURAL NUMBERS

J. LAMBEK, McGill University, and L. MOSER, University of Alberta*

1. In 1926, S. Beatty** proposed the following problem: If x is a positive irrational number, the sequences $m(1+x)$ and $n(1+x^{-1})$ jointly contain one and only one term between each pair of consecutive positive integers. (Here and elsewhere in this paper, m and n range over the set of positive integers.) This was proved by A. Ostrowski and A. C. Aitken; the latter obtained a similar result for rational x.

We shall call two sets of positive integers *complementary* if they have no common elements and together exhaust all positive integers. As usual, let $[x]$ denote the greatest integer in x. Then another way of stating the above result is to say that the sets of positive integers $[xm]+m$ and $[x^{-1}n]+n$ are complementary.

This problem has re-appeared in the literature at various times‡, in particular its connection with Wythoff's game§ has been noted. It seems however that no really wide generalization has been given, in spite of the fact that other such pairs of complementary sequences are easily constructed. Thus $[e^m]+m$

* This paper was written jointly at the Summer Research Institute of the Canadian Mathematical Congress.

** This MONTHLY, problem 3173, vol. 33, 1926, p. 159.

‡ J. V. Uspensky and M. A. Heaslet, Elementary Number Theory, New York, 1939, page 98. See also this MONTHLY, problems 4270, vol. 54, 1947, p. 549, and 4399, vol. 57, 1950, p. 343.

§ W. W. Rouse Ball and H. S. M. Coxeter, Mathematical Recreations and Essays, London, 1944, pp. 38–39. See also D. B. Sawyer, A Property of the Golden Section, Eureka, March, 1948.

and $[\log n]+n$ are complementary, and so are p_m+m and $\pi(n-1)+n$, where p_m denotes the m-th prime, and $\pi(n)$ the number of primes not exceeding n.

2. In what follows it will be convenient to use the term *number* to denote a non-negative integer or infinity. If $f=\{f(1), f(2), \cdots\}$ is a non-decreasing sequence of numbers, it must be of one of three types:

(A) $f(n)$ is finite for a finite number of n only,

(B) $f(n)$ is finite for all n and constant from a certain n on,

(C) $f(n)$ is finite for all n but tends to infinity with n.

We then define

$$f^{\dagger}(n) = \text{number of } m \text{ such that } f(m) < n. \tag{1}$$

It is easily verified that f is of type A, B, or C if and only if $f^{\dagger}$ is of type B, A, or C respectively.

Two sequences f and g of numbers will be called *inverses* provided, for all pairs of positive integers m, n,

$$f(m) < n \text{ or } g(n) < m, \text{ but not both.} \tag{2}$$

Note that the inequality $f(m)<n$ is false when $f(m)$ is infinite. We shall prove

THEOREM 1. *A sequence of numbers has an inverse if and only if it is non-decreasing. The inverse of f is unique and coincides with $f^{\dagger}$.*

Proof. Assume f has an inverse g. If g' were another inverse, (2) would imply that $g(n)<m$ if and only if $g'(n)<m$, for all positive integers m, n. If $g(n)$ is finite, take $m=g(n)+1$. Then $g'(n)<g(n)+1$ so that $g'(n)\leqq g(n)$ is also finite. Similarly, if $g'(n)$ is finite, then $g(n)\leqq g'(n)$. Hence, in any case, $g(n)=g'(n)$. Thus g is unique.

Next we wish to show that $f(m+1)\not< f(m)$. Without loss in generality we may assume that $f(m+1)$ is finite, say $f(m+1)=n-1<n$. Then, by (2), $g(n)\not< m+1$, and *a fortiori* $g(n)\not< m$. Again by (2), $f(m)<n$, so that $f(m)\leqq n-1 = f(m+1)$. Thus f is non-decreasing.

Conversely let f be non-decreasing, so that $f^{\dagger}$ is defined by (1). Consider the inequality $f^{\dagger}(n)<m$. According to (1), this means that the inequality $f(x)<n$ has fewer than m solutions for x. Since f is non-decreasing, $x=m$ is a solution if and only if all positive integers $x\leqq m$ are solutions, that is, if and only if the number of solutions is m or more. Thus $f^{\dagger}(n)<m$ if and only if $f(m)\not< n$. In view of (2), $f^{\dagger}$ is inverse to f.

This completes the proof of theorem 1. We may remark in passing that $(f^{\dagger})^{\dagger}=f$, as follows immediately from theorem 1 and the symmetry of definition (2). We shall consider some examples of pairs of inverse sequences.

Example 1. $f(m)=p_m$, $f^{\dagger}(n)=\pi(n-1)$. In general, if $f(m)=m$-th largest positive integer with property P, or $f(m)=$ infinity if fewer than m positive integers have property P, then $f^{\dagger}(n)=$ number of positive integers less than n with property P.

Example 2. $f(m)=[e^m]$, $f^\dagger(n)=[\log n]$; $f(m)=m^2$, $f^\dagger(n)=[\sqrt{n-1}]$. In general, let $\phi(x)\geqq 0$ be any real-valued strictly increasing function defined for all real $x\geqq 0$, $f(m)=[\phi(m)]$, then $f^\dagger(n)=$number of positive integers less than $\phi^{-1}(n)$. (If $\phi^{-1}(n)$ is undefined, then $f^\dagger(n)=0$.)

Example 3. $\pi(m^2)$ and $[\sqrt{p_n}]$ are inverses. In general, if f and g are non-decreasing sequences of positive integers, then $f^\dagger(g(m))$ and $g^\dagger(f(n)+1)$ are inverses.

3. If F is a set of positive integers, let $F(m)$ denote the m-th largest member of F, as long as m does not exceed the number of elements of F, otherwise let $F(m)$ denote infinity. The equation $F(m)=f(m)+m$ establishes a one to one correspondence between all non-decreasing sequences f of numbers and all sets F of positive integers, as is easily verified. (It is understood that infinity plus or minus a finite number is infinity.) This correspondence will be exploited in the following

THEOREM 2. *Two non-decreasing sequences f and g of numbers are inverses if and only if the corresponding sets F and G of positive integers, defined by $F(m)=f(m)+m$ and $G(n)=g(n)+n$, are complementary.*

Proof. Let f and g be inverses. By (2), the inequalities $f(m)<n$ and $g(n)<m$ are contradictories, in the sense that one and only one of them is true. Hence also the inequalities $F(m)<m+n$ and $G(n)<m+n$ are contradictories, and we conclude that $F(m)$ is never equal to $G(n)$. In particular, the set of all $F(m)$ with $m\leqq a$ and the set of all $G(n)$ with $n\leqq f(a)$ have no elements in common. Let $f(a)$ be finite, then none of the elements in either of these sets exceeds $a+f(a)$; for we infer from (2) that $g(f(a))<a$ since $f(a)\not<f(a)$. Thus the two sets just considered contain together the first $a+f(a)$ positive integers. If f is not of type A, this argument holds for any a, hence the sets F and G are complementary. If f is of type A, then g is of type B, and the same argument can be used with f and g interchanged.

Conversely, let F and G be complementary sets. Then f is a non-decreasing sequence of numbers, hence possesses an inverse $f^\dagger$, by theorem 1. By the above, $f^\dagger(n)+n$ defines the complementary set of F, hence coincides with $g(n)+n$. Thus $g=f^\dagger$ is the inverse of f.

This completes the proof of theorem 2.

Example 4. We wish to calculate the n-th non-square. Let $F(m)=m^2$ be the m-th square. Then $f(m)=m^2-m$ and $f^\dagger(n)=$number of m such that $m^2-m<n$, which is the same as the largest integer m satisfying this inequality. Since both sides of the inequality are integers, we may just as well compute the largest integer m satisfying $m^2-m+1/4<n$, that is $m-1/2<\sqrt{n}$. Thus $f^\dagger(n)=[1/2+\sqrt{n}]=\{\sqrt{n}\}$, where $\{x\}$ denotes the closest integer to x. The n-th non-square is therefore $n+\{\sqrt{n}\}$.

Example 5. In the same way we find that the n-th non-triangular number is $n+\{\sqrt{2n}\}$.

4. If F is a set of positive integers, let CF denote the complementary set. We have shown in theorem 2 that $CF(n) = n + f^{\dagger}(n)$, where $f(m) = F(m) - m$. The set F may be identified with the sequence $\{F(1), F(2), \cdots\}$, which is a non-decreasing sequence of numbers, and therefore possesses an inverse $F^{\dagger}$. The question arises: Can CF be expressed in terms of $F^{\dagger}$ rather than $f^{\dagger}$? It turns out to be more convenient, if instead of $F^{\dagger}$ we introduce F^{*} defined as follows:

$$(2') \qquad F^{*}(n) = F^{\dagger}(n+1) = \text{number of elements of } F \text{ not exceeding } n.$$

This may be supplemented by defining $F^{*}(\infty) = \infty$. We shall need the following

LEMMA 1. $n + F^{*}(CF(n)) = CF(n)$.

Proof. When $CF(n)$ is infinite, this is clear. Otherwise $F^{*}(CF(n))$ is the number of elements of F not exceeding $CF(n)$. Now the number of all positive integers not exceeding $CF(n)$ is $CF(n)$, and the number of elements of CF not exceeding $CF(n)$ is n. The result follows.

We shall define recursively

$$(3) \qquad F_0(n) = n; \qquad F_k(n) = n + F^{*}(F_{k-1}(n)) \qquad \text{for } k > 0.$$

Then we obtain the following

THEOREM 3. $CF(n) = \lim_{k\to\infty} F_k(n)$.

Proof. We first show that $F_{k+1}(n) \geqq F_k(n)$. Clearly this is true for $k = 0$. Suppose $k > 0$ and $F_k(n) \geqq F_{k-1}(n)$; then

$$F_{k+1}(n) = n + F^{*}(F_k(n)) \geqq n + F^{*}(F_{k-1}(n)) = F_k(n),$$

by (3), induction hypothesis, and the fact that F^{*} is non-decreasing.

We next show that $F_k(n) \leqq CF(n)$. Without loss in generality we may assume that $CF(n)$ is finite. Suppose $F_{k-1}(n) \leqq CF(n)$, then

$$F_k(n) = n + F^{*}(F_{k-1}(n)) \leqq n + F^{*}(CF(n)) = CF(n),$$

by (3), induction hypothesis, monotoneness of F^{*}, and lemma 1.

Thus we may write $G(n) = \lim_{k\to\infty} F_k(n) \leqq CF(n)$. Without loss in generality we may assume that $G(n)$ is finite. Then for sufficiently large k,

$$G(n) = F_{k-1}(n) = F_k(n) = n + F^{*}(F_{k-1}(n)) = n + F^{*}(G(n)),$$

so that $n = G(n) - F^{*}(G(n))$. By definition, this is the number of elements of CF not exceeding $G(n)$. Thus n is the number of m such that $CF(m) \leqq G(n)$, whence $CF(n) \leqq G(n)$. Since we have already shown the converse inequality, we have equality, as was to be proved.

Example 5. The n-th non-prime is the limit of the sequence: n, $n + \pi(n)$, $n + \pi(n + \pi(n))$, $\cdots$.

5. For given n, the sequence $F_k(n)$, $(k = 0, 1, \cdots)$, is a sequence of integers, hence attains its limit $CF(n)$ in a finite number of steps, provided this

limit is finite. In fact, it follows from theorem 3 that we need not go beyond $k = CF(n) - n$. It may happen that the number of steps need not depend on n. We shall see that in many cases two steps are sufficient.

THEOREM 4. *If* $F(m+1) - F(m) \geqq m$, *then* $CF(n) = F_2(n)$.

Proof. In view of lemma 1 and (3), it suffices to prove that $F^*(CF(n)) = F^*(F_1(n))$. We know from the proof of theorem 3 that $F_1(n) \leqq CF(n)$, hence we need only show that $F^*(CF(n)) \leqq F^*(F_1(n))$, which means that any element of F not exceeding $CF(n)$ does not exceed $F_1(n)$. Assuming $F(m) \leqq CF(n)$, we shall prove that $F(m) \leqq F_1(n)$.

Since F and CF have no elements in common, we even have $F(m) < CF(n)$, so that $(CF)^*(F(m)) \leqq n-1$, hence by lemma 1,

$$F(m) = m + (CF)^*(F(m)) \leqq m + n - 1. \tag{4}$$

If $m = 1$, we deduce immediately that $F(1) \leqq n \leqq F_1(n)$, by (3). There remains the case $m > 1$.

If $m > 1$, by hypothesis and (4), $F(m-1) + m - 1 \leqq F(m) \leqq m + n - 1$, so that $F(m-1) \leqq n$, whence $m - 1 \leqq F^*(n)$. Hence, by (4) again,

$$F(m) \leqq n + m - 1 \leqq n + F^*(n) = F_1(n),$$

as was to be proved.

Example 6. The n-th positive integer which is not a perfect k-th power ($k \geqq 2$) is $n + [(n + [n^{1/k}])^{1/k}]$. For $k = 2$ this gives a new formula for the n-th non-square.

Example 7. The n-th positive integer not of the form $[e^m]$ with $m \geqq 1$ is $n + [\log (n+1+[\log (n+1)])]$.

Postscript. It has come to the authors' attention that Viggo Brun [*Rechenregel zur Bildung der n-ten Primzahl*, Norsk. Mat. Tidsskr., vol. 13, 1931, pp. 73–79] proved the following: If $n_k = n - \pi(n + n_1 + \cdots + n_{k-1})$ for $k > 0$, *then* $n + n_1 + \cdots + n_k \to p_n$ *as* $k \to \infty$. This result is an immediate consequence of Theorem 3. For, letting $CF(n) = p_n$, one obtains $F^*(n) = n - \pi(n)$ and $n + n_1 + \cdots + n_k = F_k(n) \to p_n$. D. H. Lehmer [*An inversive algorithm*, Bull. Amer. Math. Soc., vol. 38, 1932, pp. 693–694] generalized Brun's result from p_n to any increasing sequence of positive integers, thus obtaining a result essentially equivalent to Theorem 3.

CARNOT AND THE CONCEPT OF DEVIATION

C. B. BOYER, Brooklyn College

In this MONTHLY for October, 1952 (p. 535), one reads that "The concept of aberrancy . . . was originally introduced by Transon (using the name *deviation*)" [1]. It is the intention here to point out that this notion is older still, for it appeared in 1803 in the *Géométrie de position* of Lazare Nicholas Marguerite Carnot, published thirty-eight years before Transon wrote. The reputation of this book has been well-known, yet references to it leave the impression that it is devoted exclusively to synthetic geometry. It is indeed true that in it the author contributed significantly to so-called "pure" geometry; but the last of the five main sections of the work is concerned primarily with the use of analytic methods. In fact, the point of view toward coordinates here presented is the broadest of any mathematician since Newton, another who traditionally is placed among the synthesists. Carnot may well have been unaware of the suggestion of polar, bipolar, and other coordinate systems in Newton's *Methodus fluxionum*, for this has been overlooked by other mathematicians and historians [2] just as has a similar contribution in the *Géométrie de position*. Under the heading, "*De la détermination d'un point dans l'espace, et du changement de ses coordonnées*," Carnot proposed a wide variety of systems: polar, bipolar, biangular, and various triangular types. In such cases the equations of a curve depend on the elements used as a frame of reference; and so Carnot looked for intrinsic coordinates. He wrote,

> For any curve there exists at each point a certain line which does not depend on any particular hypothesis or any basis of comparison taken in absolute space; this is the radius of curvature . . . I believe nevertheless that it is possible to find various variables which shall have the condition demanded.

Carnot considered using arc length as a second coordinate, along with the radius of curvature; but he rejected this because it depends upon the choice of a fixed point from which the distance is measured. Instead he introduced what now is known as the angle of deviation—or, strictly speaking, its complement:

> I propose, for example, the angle which is formed, at the point describing the curve, by the tangent and the line which bisects the infinitesimally small secants drawn through the curve parallel to this tangent.

In illustrating this idea, he replaced the tangent by the normal, thus bringing his angle into conformity with the modern aberrancy. If M is a point on a curve and K is the center of curvature for this point, Carnot drew perpendicular to MK a variable secant mm' with midpoint n. His angle of deviation was then the angle between MK and the limiting position of the line Mn as m and m' approach M along the curve. As an example of the use of the angle as a coordinate, he cites the parabola $y^2 = px$, for which one finds $r = \frac{1}{2}p \csc^3 z$, where r is the radius of curvature and z the angle of deviation. This latter equation, in

curvature and aberrancy as coordinates, is one form of intrinsic equation of the parabola. Carnot also derived a general formula for cot z, pointing out that for a circle this expression vanishes; *i.e.*, in modern terminology, the deviation of a circle at every point is zero [3].

In 1841 Transon gave a formula for tan z, introducing the phrase "axis of deviation" and pointing out that the deviation is a geometric representation of the third derivative. He also called attention to some remarkable properties of the deviation of conics, including the relation $\tan z = r'/r$, where r and r' are the radii of curvature of the conic and its evolute; and he introduced the center of deviation [4]. Walker recently has added striking relations between the deviation of a conic and its differential equation [5]. The concept which Carnot introduced so unobtrusively thus has led to developments comparable to those connected with the idea of curvature; yet the adumbration in the *Géométrie de position* has been so completely overlooked that there seems to have been no reference to it by mathematicians or historians. Even Transon himself, *ancien élève* of the École Polytechnique which Carnot helped to establish, made no mention of it.

The year 1953 marks the bicentennial of the birth of Lazare Carnot (as well as the sesquicentennial of his *Géométrie de position*), and hence brief attention may appropriately be called to the extraordinary career of this versatile mathematician. He was born at Nolay (Côte-d'Or), on May 13, 1753, one of eighteen children, several of whom came to occupy positions of prominence. Lazare in 1771 entered the École de Mézières, where he studied under Monge. His fellow military students regarded him as *un original* because he preferred the library to the cafe. His early published works were both literary and scientific, for he wrote considerable poetry and took a particular interest in balloons, fortifications, and politics. When the Revolution opened, Carnot, then a captain, accepted enthusiastically the views of the National Assembly; but he took no active role until in 1791 he was elected to the Legislative Assembly. Subsequently designated to the National Convention, he was among those who voted for the execution of Louis XVI on grounds of "justice and policy alike." Becoming more and more deeply involved in politics, he was chosen a member of the Committee of Public Safety in 1793. Charged with the organization and operation of the revolutionary army, his genius achieved such striking success against the invading forces that he was honored spontaneously as "The Organizer of Victory." Carnot did not hesitate to seize a musket and lead his troops against the enemy. Robespierre was bitterly jealous of his military ability and St. Just accused him of moderantism, but he survived the Terror through his indispensability to the defence of France. However, he was proscribed in 1797 as the result of false charges of implication in a royalist plot. His name was stricken from the rolls of the *Institut*, and his chair as geometer was filled, *mirabile dictu*, by General Bonaparte. Monge, the great geometer, was so mesmerized by Napoleon that even he joined in this unanimously approved outrage against mathematical honesty.

When Bonaparte scored his *coup d'Etat* of 1799, Carnot returned to France to become minister of war and was reelected to the *Institut*. He was elected a member of the Tribunate, and, being an inflexible republican, he exerted all his influence against Napoleon's imperial designs. In the midst of extraordinary factional turmoil, Carnot remained aloof from political parties, opposing tyranny of every description. A man of courage and conviction, in 1804 he delivered a brilliant speech against the conferring of the Imperial Dignity upon Napoleon [6], but his was the lone negative vote. Yet, when in 1814 Carnot saw France in jeopardy, he offered Napoleon the services of his "weak sexagenarian arm." As commander at Antwerp he held out against the enemy with notorious stubbornness, withdrawing only after the abdication of the emperor. Carnot did not conspire to recall Napoleon; but during the Hundred Days he again served his country, this time as Minister of the Interior, and he struggled in vain against the second abdication of Napoleon. He became a member of the provisional government, but he was exiled on the return of Louis XVIII and died at Magdeburg on Aug. 2, 1823 [7]. Carnot's influence, however, was felt long afterward: through his memory, in his descendants, and through his published works. His eldest son was the famous physicist, Sadi Carnot, author of *Réflexions sur la puissance motrice du feu* (1824); and a grandson, Marie François Sadi Carnot, became the fourth president of the Republic.

Notwithstanding a life of intense military and political activity, Carnot contributed regularly to mathematics. The work which enjoyed greatest popularity was his *Réflexions sur la métaphysique du calcul infinitésimal* of 1797. In this he sought to reconcile the various approaches to the calculus—the method of exhaustion, Leibnizian differentials, Newtonian fluxions, the limits of D'Alembert, the series of Lagrange, and others—concluding, unfortunately, that the true unifying basis was to be found in the principles of the compensation of errors. The *Réflexions* appeared in half a dozen French editions, as well as in many other languages, including English [8]; but as a permanent contribution to mathematics it falls far below the *Géométrie de position* of 1803.

Monge and Carnot may be looked upon as the two founders of modern synthetic geometry, the former through his influence as a great teacher at the École Polytechnique, the latter through his published works, especially the *Géométrie de position*. This contains numerous theorems of pure geometry, a striking example of which may be cited here: If M, N, P, Q are the areas of the four faces of a tetrahedron, and if m, n, p are the angles between the faces N and P, M and P, and M and N respectively, then $Q^2 = M^2 + N^2 + P^2 - 2MN \cos p - 2NP \cos m - 2PM \cos n$ [9]. Yet the best known of Carnot's contributions to pure geometry comes not from the *Géométrie de position*, but from his *Essai sur le théorie des transversales* of 1806. This contains the celebrated "theorem of Carnot"—Given any algebraic curve of order n which cuts a triangle ABC, let A_1 be the product of the n distances, real or imaginary, from A to the n points of intersection of the curve with the side AB, and let B_1 and C_1 be defined similarly for the sides BC and CA; and let A_2, C_2, and B_2 be the simi-

lar products corresponding to the sides AC, CB, and BA respectively. Then $A_1B_1C_1 = A_2B_2C_2$. If the curve is a straight line, one has a well-known theorem of antiquity; if the curve is a cubic, it follows from Carnot's theorem that the three points of inflection lie on a straight line, a familiar result of the eighteenth century.

Carnot was aware that one of the advantages of analytic geometry was its generality; and hence he had sought, in *De la corrélation des figures de géométrie* (1801), to establish for synthetic geometry a comparable universality. Here he proposed a unifying idea somewhat akin to the controversial "principle of continuity" of Poncelet. This idea was further developed in the *Géométrie de position.*

Among Carnot's earliest works was the *Essai sur les machines en général* (1783), which contained the Carnot theorem on the loss of energy due to abrupt changes in velocity. This work, revised, appeared again in 1803 as *Principes fondamentaux de l'équilibre.*

There have been greater scientists and geometers than Carnot; there may have been greater political and military figures (although none more upright and courageous). But as a brilliant example of a man of thought and of action, it is difficult indeed to find a mathematician who outshines Carnot the elder—"The Great Carnot."

References

1. This ascription is found in other sources also: George Salmon, Higher Plane Curves, 3rd ed., London, 1879, pp. 368–369; E. J. Wilczynski, Some remarks on the historical development and the future prospects of the differential geometry of plane curves, Bull. Amer. Math. Soc. vol. 22, 1915–1916 pp. 320–324 (especially p. 321); Encyklopädie der Mathematischen Wissenschaften, III, vol. 3, p. 40.

2. See Newton as an originator of polar coordinates, this MONTHLY, vol. 56, 1949, pp. 73–78.

3. Géométrie de position, Paris, 1803, pp. 475 f.

4. Abel Transon, Recherches sur la courbure des lignes et des surfaces, Journal de mathématiques pures et appliquées, vol. 6, 1841 pp. 191–208.

5. This MONTHLY, vol. 59, 1952, pp. 531–539.

6. For English translations of some of Carnot's famous speeches, see Lewis Goldsmith, A sketch of M. Carnot's life, together with some remarkable speeches which he made, London, c. 1814, and An exposition of the political conduct of Lieut. General Carnot, since the first of July, 1814. By himself. To which is added Carnot's speech in the Tribunate in 1804, trans. by Henry Wheaton, New York, 1815).

7. There is an English translation of Arago's standard scientific biography. See D. F. J. Arago, Biographies of distinguished scientific men (2nd series, Boston, 1859). A brief note is also found in D. E. Smith, Lazare Nicholas Marguerite Carnot, *Scien. Monthly*, vol. 37, 1933, pp. 188–189. For a full-length biography in English, but largely from the political point of view, see Huntly Dupre, Lazare Carnot. Republican patriot, Oxford, Ohio, 1940. There are several recent French biographies, including Henri Carré, Le grand Carnot, vol. I, Paris, c. 1950, as well as an older German account, K. Fink, Lazare Nicolas Marguerite Carnot, sein Leben und seine Werke, Tübingen, 1894. A chapter on Carnot, with particular reference to this work on fortifications, is included in E. M. Lloyd, Vauban, Montalembert, Carnot: Engineer Studies, London, 1887, pp. 154–199.

8. For editions and further analysis of the work see my Concepts of the Calculus, New York, 1939, pp. 257 f.

9. See Niels Nielsen, Géomètres français sous la revolution, Copenhagen, 1929, for further details on his life and work. Cf. also Maximilien Marie, Histoire des sciences mathématiques et physiques, 12 vols., Paris, 1883–1888, X, pp. 157–168.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

INEQUALITIES CONNECTED WITH DEFINITE HERMITIAN FORMS, II

A. OPPENHEIM, University of Malaya, Singapore

1. Let $A=(a_{ik})$, $B=(b_{ik})$ be two non-negative definite Hermitian matrices of order n. Then by considering the corresponding non-negative definite quadratic forms we see plainly that the matrix

$$C=(c_{ik}), \qquad c_{ik}=a_{ik}+b_{ik} \tag{1}$$

is also non-negative definite Hermitian.

THEOREM 1. *Suppose that the latent roots of A, B, C are the (non-negative) numbers arranged in increasing order:*

$$\alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_n, \qquad \beta_1 \leqq \beta_2 \leqq \cdots \leqq \beta_n, \qquad \gamma_1 \leqq \gamma_2 \leqq \cdots \leqq \gamma_n. \tag{2}$$

Then

$$(\gamma_1\gamma_2\cdots\gamma_i)^{1/i} \geqq (\alpha_1\cdots\alpha_i)^{1/i}+(\beta_1\cdots\beta_i)^{1/i}, \qquad (i=1,2,\cdots,n). \tag{3}$$

If we take $i=n$ this inequality becomes (if $a=\det A$, *etc.*).

$$c^{1/n} \geqq a^{1/n}+b^{1/n}, \tag{4}$$

an inequality due (for real forms) to Minkowski [2] from which as I have shown [3] inequalities of Hadamard and Fischer are deducible. (See also Hardy, Littlewood, and Pólya, *Inequalities.*)

Suppose that λ and μ are arbitrary non-negative numbers whose sum is unity. Then from (4) we obtain

$$\det(\lambda a_{ik}+\mu b_{ik}) \geqq (\lambda a^{1/n}+\gamma b^{1/n})^n, \tag{5}$$

and so by the inequality of the arithmetic mean and geometric mean,

$$\det(\lambda a_{ik}+\mu b_{ik}) \geqq (\det(a_{ik}))^{\lambda}(\det(b_{ik}))^{\mu}. \tag{6}$$

This inequality was given recently by Ky Fan [1]. Plainly (5) is stronger than (6).

2. The proof of Theorem 1 depends on an interesting algebraic inequality which is worth stating as a theorem.

THEOREM 2. *Suppose that the numbers e_i, f_i $(i=1, \cdots, n)$ are non-negative. Suppose that e_i^*, f_i^* are the same sets arranged in non-decreasing order. Then if*

$$\prod^* = (e_1^* + f_n^*)(e_2^* + f_{n-1}^*) \cdots (e_n^* + f_1^*), \tag{7}$$

and

$$\prod_* = (e_1^* + f_1^*)(e_2^* + f_2^*) \cdots (e_n^* + f_n^*), \tag{8}$$

$$\prod^* \geqq \prod_{i=1}^{n} (e_i + f_i) \geqq \prod_*. \tag{9}$$

Equality holds throughout (9) *if one of the sets consists of equal numbers. Equality holds on the right of* (9) *if one of the permutations which carries (e_i) into (e_i^*) carries (f_i) into (f_i^*). Equality holds on the left if a permutation carrying (e_i) into (e_i^*) carries (f_i) into the reverse set to (f_i^*).*

The second inequality in (9) is a special case of an inequality of Ruderman [4].

We may suppose without loss of generality that $e_i^* = e_i$ for each i. This is merely a renumbering.

For $n=1$ the theorem is trivial. For $n=2$ it follows from the obvious identity

$$(x + y')(x' + y) - (x + y)(x' + y') = (x' - x)(y' - y)$$

when we take

$$0 \leqq x \leqq x', \qquad 0 \leqq y \leqq y'.$$

Consider the product

$$\prod = (e_1 + f_1)(e_2 + f_2) \cdots (e_n + f_n), \qquad (n \geqq 3).$$

If in any pair (e_i+f_i) $(e_{i+1}+f_{i+1})$ we have unequal f_i, f_{i+1}, then by the case $n=2$ we get the minimum if $f_i<f_{i+1}$ and the maximum if $f_i>f_{i+1}$. Thus after a finite number of steps we reduce $\prod$ to $\prod_*$ by interchanges which do not increase $\prod$ and likewise we can change $\prod$ to $\prod^*$ by interchanges which do not decrease $\prod$. The maximum number of steps needed will be $\frac{1}{2}n(n-1)$. It is clear that Theorem 2 follows at once.

3. Suppose that A is positive definite. There is a non-singular transformation which carries A and B into diagonal form with elements α_i (in some order) and β_i (in some order). This transformation carries C into the diagonal form γ_i (in some order) so that

$$\gamma_i = \lambda_i + \mu_i$$

where the λ_i are a permutation of the α_i and the μ_i are a permutation of the β_i.

By Theorem 2

$$\gamma_1\gamma_2 \cdots \gamma_i \geqq \prod_{j=1}^{i} (\lambda_j^* + \mu_j^*) \geqq \prod_{j=1}^{i} (\alpha_j + \beta_j),$$

since $\lambda_1^*, \cdots, \lambda_i^*$ are a selection (in non-decreasing order of magnitude) from the non-decreasing set $\alpha_1, \cdots, \alpha_n$ so that

$$\lambda_j^* \geqq \alpha_j, \quad \text{and likewise} \quad \mu_j^* \geqq \beta_j.$$

Now apply Hölder's inequality to obtain

$$(\gamma_1\gamma_2 \cdots \gamma_i)^{1/i} \geqq \left(\prod_1^i \alpha_j\right)^{1/i} + \left(\prod_1^i \beta_j\right)^{1/i}.$$

Theorem 1 follows therefore when A is positive definite, B non-negative definite. But if A and B are both non-negative and neither is positive definite, there is nothing to prove.

If we apply (3) to the matrix $\lambda A + \mu B$ where λ and μ are positive numbers with sum unity and then use the inequality of the arithmetic mean and the geometric mean we obtain

THEOREM 3. *Let $\gamma_1 \leqq \gamma_2 \leqq \cdots \leqq \gamma_n$ be the latent roots of the matrix $C = \lambda A + \mu B$ $\lambda \geqq 0$, $\mu \geqq 0$, $\lambda + \mu = 1$, where A and B are non-negative Hermitian matrices with latent roots*

$$\alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_n, \qquad \beta_1 \leqq \beta_2 \leqq \cdots \leqq \beta_n;$$

then, for $i = 1, 2, \cdots, n$,

$$\gamma_1\gamma_2 \cdots \gamma_i \geqq (\alpha_1\alpha_2 \cdots \alpha_i)^{\lambda}(\beta_1\beta_2 \cdots \beta_i)^{\mu}.$$

Theorem 3 is due to Ky Fan (Problem 4430, this MONTHLY, proposed in vol. 58, 1951, p. 194; solution in vol. 60, 1953, p. 50). The inequality is weaker than that of Theorem 1.

4. I am indebted to the referee for giving an alternative proof of Theorem 1 which employs another result of Ky Fan (Problem 4429, *ibid.*, proposed in vol. 58, 1951, p. 194; solution in vol. 60, 1953, p. 48).

"Let A, B be two non-negative Hermitian matrices of order n. Let the eigenvalues of A, B, $A+B$ be

$$\begin{aligned} A:&\ \alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_n, \\ B:&\ \beta_1 \leqq \beta_2 \leqq \cdots \leqq \beta_n, \\ A+B:&\ \gamma_1 \leqq \gamma_2 \leqq \cdots \leqq \gamma_n, \end{aligned}$$

respectively. Let $x_1, \cdots, x_n$ be n orthonormal eigenvectors of $A+B$ such that

$$(A+B)x_j = \gamma_j x_j \qquad (1 \leqq j \leqq n).$$

According to the theorem in Problem 4429 we have

$$\alpha_1\alpha_2 \cdots \alpha_i \leqq \prod_{j=1}^{i} (Ax_j, x_j), \qquad \beta_1\beta_2 \cdots \beta_i \leqq \prod_{j=1}^{i} (Bx_j, x_j),$$

and therefore

$$(\alpha_1\alpha_2 \cdots \alpha_i)^{1/i} + (\beta_1\beta_2 \cdots \beta_i)^{1/i} \leqq \left[\prod_{j=1}^{i} (Ax_j, x_j)\right]^{1/i} + \left[\prod_{j=1}^{i} (Bx_j, x_j)\right]^{1/i}.$$

By Hölder's inequality, the right side is not greater than

$$\left[\prod_{j=1}^{i} ((A + B)x_j, x_j)\right]^{1/i} = (\gamma_1\gamma_2 \cdots \gamma_i)^{1/i}.$$

Hence

$$(\alpha_1\alpha_2 \cdots \alpha_i)^{1/i} + (\beta_1\beta_2 \cdots \beta_i)^{1/i} \leqq (\gamma_1\gamma_2 \cdots \gamma_i)^{1/i}."$$

But the proof of Theorem 1 given in Section 2 does not use the theorem of Problem 4429.

References

1. Ky Fan, On a theorem of Weyl concerning eigenvalues of linear transformations, I, II, Proc. Nat. Acad. Sci., vol. 35, 1949, pp. 652–655, vol. 36, 1950, pp. 31–34. (I have not been able to get access to these papers.)

2. H. Minkowski, Diskontinuitätsberich für arithmetisch Äquivalenz, Journ. für Math., vol. 129, 1905, pp. 220–274.

3. A. Oppenheim, Inequalities connected with definite Hermitian forms, Journ. London Math. Soc., vol. 5, 1930, pp. 114–119.

4. H. D. Ruderman, Two new inequalities, this MONTHLY, vol. 59, 1952, pp. 29–32.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

REMARK ON A GRAPHICAL PROCEDURE

W. T. GUY, JR., The University of Texas

In a recent note, Russell [1] extended the graphical method of Liénard [2, 3] (used in plotting phase trajectories in non-linear problems) to some differential equations of the type

$$\frac{dy}{dx} = \frac{\phi(y) + x}{\psi(x) + y}. \tag{1}$$

Russell's construction is based on the curves Γ_1: $[x=-\phi(y)]$ and Γ_2: $[y=-\psi(x)]$, where $\phi(y)$ and $\psi(x)$ denote continuous functions.

The purpose of this remark is to point out that his construction, with a slight modification, is also valid for certain differential equations of the type

$$\frac{dy}{dx}=\frac{g(x,\ y)}{h(x,\ y)}. \tag{2}$$

We assume $g(x,\ y)=0$ defines x as a continuous (single-valued) function of y [say, $x=-\phi(y)$] and $h(x,\ y)=0$ defines y as a continuous (single-valued) function of x [say, $y=-\psi(x)$] over the required domains. Hence, $g(x,\ y)=[x+\phi(y)]\cdot c(x,\ y)$, $h(x,\ y)=[y+\psi(x)]k(x,\ y)$, and equation (2) becomes

$$\frac{dy}{dx}=\frac{\phi(y)+x}{\psi(x)+y}\,\frac{c(x,\ y)}{k(x,\ y)}=\frac{\phi(y)+x}{\psi(x)+y}\,R(x,\ y). \tag{3}$$

Here $R(x,\ y)$ might be called a rotation factor. The construction of Russell based on his curves Γ_1 and Γ_2 is repeated except now the line PD must be replaced by a line PE whose slope is equal to that of PD when multiplied by the value of $R(x,\ y)$ at point P.

The justification and limitations of the method are clear. If $R(x,\ y)=1$, the Russell construction obtains. If $R(x,\ y)=$a constant, the indicated modification is easily carried out. A protractor graduated with numbers indicating slopes will be of help in measuring the slope of PD and in drawing the line PE, especially for the case $R(x,\ y)\neq$ a constant. This procedure has some merit over the usual directional field scheme, even in this latter case, if Γ_1 and Γ_2 can be easily determined and if $R(x,\ y)$ is simpler to use than $g(x,\ y)/h(x,\ y)$.

References

1. J. P. Russell, A graphical procedure, this MONTHLY, vol. 60, 1953, p. 710.
2. A. Liénard, Etude des oscillations entretenues, Rev. Gén. d'Elect., 1928.
3. J. J. Stoker, Non-linear Vibrations in Mechanical and Electrical Systems, Interscience, New York, 1950, pp. 31–36.

COMMENTS ON THE POLAR PLANIMETER

L. I. LOWELL, Northrop Aircraft

Figure 1 shows a popular design of planimeter used in engineering offices for measuring areas of various shapes. No doubt the question has occurred to many operators: "Is the total angle generated by the indicating roller, as the point traces a closed curve, in direct proportion to the area encompassed by the curve?"

The following verification of the planimeter would seem worthwhile, first, because it is more straightforward and complete than the brief statement in Calculus books that "the polar planimeter is an instrument which makes use of

Guldin's Formula for measuring areas." Secondly, it provides a valuable example of a curvilinear coordinate system and the teacher of Advanced Calculus is frequently hard put for examples when covering that topic.

For purposes of analysis the planimeter will be thought of as a coordinate system. The arms have arbitrary lengths, a and b, which are constant, but can be assumed long enough for coverage of any arbitrary area. The variables, α and β, may be related to the cartesian variables, x and y, by means of the relations:

$$x = a\cos\alpha + b\cos\beta \tag{1}$$

$$y = a\sin\alpha + b\sin\beta. \tag{2}$$

The differential element of area is:

$$dA = |J|\, d\alpha d\beta \tag{3}$$

in which

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial \alpha} & \dfrac{\partial y}{\partial \alpha} \\ \dfrac{\partial x}{\partial \beta} & \dfrac{\partial y}{\partial \beta} \end{vmatrix}.$$

Computing the partial derivatives from the basic equations, we find that the Jacobian becomes:

$$J = -ab\sin(\alpha - \beta). \tag{4}$$

The element of area is thus found to be:

$$dA = -ab\sin(\alpha - \beta)d\alpha d\beta. \tag{5}$$

Now let the planimeter trace an element of area, ΔA, and an accounting will be made of the motion of the indicating wheel. We see from the drawing that while the point is moving from 1 to 2, α is being increased by $\Delta\alpha$ but there is no relative motion between the two arms. From 2 to 3, β is being increased by $\Delta\beta$ while α is held fixed. The remaining two sides of the curvilinear quadrilateral are traced by motions which are the exact reverse of those which moved the point from 1 to 2 and 2 to 3, respectively. The wheel movement resulting from the point's moving from 2 to 3 is equal and in the opposite direction to that which occurs as the point moves from 4 to 1. Hence, these movements cancel each other and will be neglected.

As the point moves from 1 to 2, the total wheel rotation will be some multiple of

$$a\cdot\Delta\alpha\cos(\alpha - \beta); \tag{6}$$

from 3 to 4 it will be the same multiple of

$$a\cdot\Delta\alpha\cos(\alpha - \{\beta + \Delta\beta\}). \tag{7}$$

Subtracting (7) from (6) and rearranging will yield:

$$a \cdot \Delta\alpha[\cos(\alpha - \beta)(1 - \cos\Delta\beta) - \sin\Delta\beta \sin(\alpha - \beta)]. \tag{8}$$

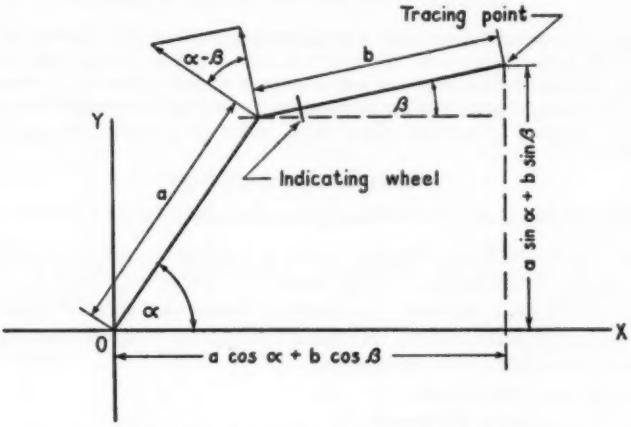


FIG. 1. The Polar Planimeter as a Coordinate System.

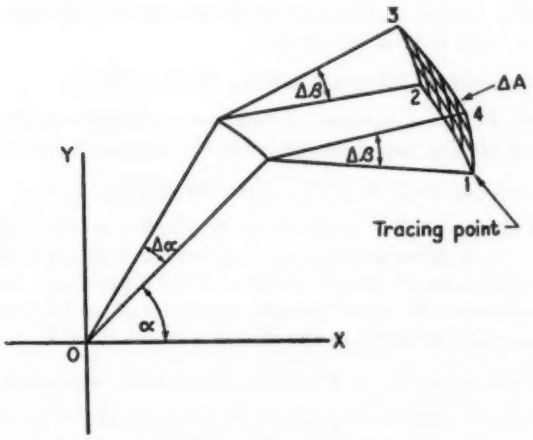


FIG. 2. Element of Area.

If we take $\Delta\alpha = d\alpha$ and $\Delta\beta = d\beta$ and neglect all powers of $\Delta\beta$ higher than the first power, we may replace (8) by

$$-a \sin(\alpha - \beta)(d\alpha d\beta)$$

which is a multiple of the right hand side of equation (5).

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1126. *Proposed by J. V. Pennington, Houston, Texas*

Smith: "Down in Todd County, which is a 23-mile square, I have a ranch—oblong in shape—measuring a whole number of miles each way."

Jones: "Hold on a minute. I happen to know the area of your ranch; let me see if I can figure its length." He figures furiously. "I need more information. Is the width more than half the length?"

Smith answers the question.

Jones: "I now know the length."

Brown: "I too know the area and, although I did not hear your answer to Jones' question, I too can tell you the length."

Green: "I did not know the area, but I can now tell you the length."

How many miles long is the ranch?

E 1127. *Proposed by Edmund DeWan, Harpur College*

Enumerate the total number of squares appearing on an $m \times n$ checker board, $m \geqq n$. Solve the analogous problem for rectangular parallelopipeds.

E 1128. *Proposed by C. S. Ogilvy, Hamilton College*

Show that the point P at eye level from which a picture high on a vertical wall subtends the maximum angle can be found as follows. With eye-level line as directrix and bottom of picture as focus, draw a parabola; from the intersection of this parabola with a line through the center of the picture parallel to the directrix, a line perpendicular to the directrix meets it at P.

E 1129. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Find an integral arithmetic progression with an arbitrarily large number of terms such that no term is a perfect rth power for $r = 2, 3, \cdots$, or N. Is this still possible if $N = \infty$?

E 1130. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let the perpendicular bisector of the median BB' of triangle ABC and the tangent at B to the circumcircle of triangle ABC cut the line AC in points M and N respectively. Show that triangle ABC is isosceles with vertex at A if and only if $AM/AN = 3/4$.

SOLUTIONS

E 1051 [1953, 551]. *Correction.* The condition given in the solution is sufficient but not necessary.

E 1088 [1954, 347]. *Correction.* In line 3 of the solution, the statement "$P_1=1$" should read "$P_1=P$."

Differential Equation of All Conics of Given Eccentricity

E 1070 [1953, 332; 1954, 50]. *Proposed by A. W. Walker, University of Toronto*

Find the lowest order differential equation satisfied by all conics with a given eccentricity e.

Solution by Alexander Oppenheim, University of Malaya. The required differential equation is

$$(8u_2)^{1/2} = k[(1 + y_1^2)(2uu_2 - u_1^2) - 4y_1u_1u^{-1/2} - 4u^{-1}], \tag{1}$$

where

$$u = y_2^{-2/3}, \qquad k = (e^2 - 1)^{1/2}/(2 - e^2), \tag{2}$$

and suffixes denote differentiations with respect to x.

Note that

$$2k = \tan \phi, \qquad e = \sec \phi/2, \tag{3}$$

where ϕ is the angle between the asymptotes of the conic.

To obtain (1) it is best to use the general equation of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{4}$$

and the relation

$$(a + b) \tan \phi = 2(h^2 - ab)^{1/2}. \tag{5}$$

From (4)

$$by = - hx - f + X^{1/2}, \qquad X = - Cx^2 - 2Gx - A, \tag{6}$$

where, as usual, $C=ab-h^2$, $G=hf-bg$, $\cdots$, are the cofactors of c, g, $\cdots$, in the determinant Δ of (4). Differentiate (6). Then

$$by_1 = - h - (Cx + G)X^{-1/2}, \tag{7}$$

$$by_2 = b\Delta X^{-3/2}, \qquad \Delta^{2/3}u = X, \tag{8}$$

by (2). Differentiate (8) twice and obtain

$$\Delta^{2/3}u_1 = - 2(Cx + G), \tag{9}$$

$$\Delta^{2/3}u_2 = - 2C, \tag{10}$$

whence, since $AC-G^2=b\Delta$,

$$\Delta^{1/3}(2uu_2 - u_1^2) = 4b. \tag{11}$$

We now employ (5) in the form

$$b(-C)^{1/2} = k(C + h^2 + b^2), \tag{12}$$

since $C=ab-h^2$. From (7), (8), (9)

$$h = -(1/4)\Delta^{1/3}(2uu_2 - u_1^2)y_1 + (1/2)\Delta^{1/3}u_1u^{-1/2}, \tag{13}$$

so that, by (10),

$$16\Delta^{-2/3}(C + h^2) = 4b\Delta^{-1/3}[(2uu_2 - u_1^2)y_1^2 - 4u_1u^{-1/2}y_1 - 4u^{-1}] \tag{14}$$

after a little reduction and by (11). From (10), (12) it follows that

$$(8u_3)^{1/2} = k[(2uu_2 - u_1^2)(1 + y_1^2) - 4u_1u^{-1/2}y_1 - 4u^{-1}], \tag{15}$$

since we may ignore $b=0$. Equation (15), of order four, is the differential equation of all conics of given eccentricity e where k and e are related by the second equation in (2).

As a check take $e=1$ or $\phi=0$. Then (15) gives $u_2=0$, the well known differential equation of all parabolas. If we put $\phi=\pi/2$, we see that the differential equation of all rectangular hyperbolas is

$$(2uu_2 - u_1^2)(1 + y_1^2) - 4u_1u^{-1/2}y_1 - 4u^{-1} = 0. \tag{16}$$

The derivative of the left hand side of (16) is $2uu_3(1+y_1^2)$, so that it is easy to verify that elimination of k from (15) leads to $u_3=0$, the differential equation for all conics. It is worth adding that (10) gives the differential equation of all conics with given area or with given product for semi-axes, since, if α, β are the semi-axes, $\alpha^2\beta^2C^3=\Delta^2$.

The Counterfeiters of Lower Slobbovia

E 1096 [1954, 46]. *Proposed by L. R. Ford, Jr., University of Illinois*

As is well known, Lower Slobbovia is too poor a country to afford its own mint. There are N coiners engaged in making Rasbuckniks, the local currency, to government specifications. However it is suspected that some of them may be counterfeiting by introducing some base metal into the alloy. Any pair of counterfeits will weigh the same, although slightly different from the weight of a good coin. Each coiner produces either all good coins or all counterfeits. With one guaranteed good coin, a set of infinitely refinable weights, a beam balance, and as many coins from each coiner as may be needed, determine in three weighings whether any of the coiners is dishonest, and which ones.

Solution by Julian Braun, China Lake, Calif. First, determine the weight of the good coin, W_g. Second, put one coin from each coiner on one side of the balance and determine their total weight, T. The discrepancy is $D=T-NW_g$. If $D=0$, all the coiners are honest. Thirdly, if $D\neq 0$, put 2^{i-1} coins from the ith coiner, $i=1, 2, \cdots, N$, on one side of the balance and determine the total weight of these coins, T'. The discrepancy in this case is $D'=T'(2^N-1)W_g$.

Find the integer S such that $D'/D=S/\beta(S)$, where $\beta(S)$ is the number of ones in the binary representation of S. Let $S=\sum_{i=1}^{N}B_i2^{i-1}$, where $B_i=0$ or 1. Then $\beta(S)$ is the number of dishonest coiners and the ith coiner is honest or dishonest according as $B_i=0$ or 1.

Also solved by G. B. Charlesworth, Octave Levenspiel, P. B. Maggs and D. B. Mumford (jointly), R. H. Marquis, D. C. B. Marsh, L. V. Mead, C. S. Ogilvy, C. R. Perisho, H. W. Resnikoff, L. A. Ringenberg, Thomas Saaty, A. J. Tingley, and the proposer. These solutions showed interesting variations from one another.

A 3:4:5 Right Triangle

E 1097 [1954, 46]. *Proposed by Leon Bankoff, Los Angeles, Calif.*

Arcs AB and CD are quadrants of circles tangent externally at their midpoints, E, and such that AC and BD when extended meet perpendicularly in F. A circle is inscribed in the mixtilinear triangle EDB, touching ED in M. G is the projection of M upon EF. Show that triangle MGF is a 3:4:5 right triangle.

Solution by W. J. Cherry, North Riverside, Illinois. Let K be the center of arc AB. Taking BF and BK as x and y axes, denote the incenter of the mixtilinear triangle by $P:(x, y)$. Let N and H be the projections of P on BF and BK. Then, by applying the Pythagorean Theorem to triangles KHP and NPF, we find

$$x^2 = 4r_1y \quad \text{and} \quad (x - r_1)^2 = r_2(2y + r_2),$$

where r_1 and r_2 denote the radii of arcs AB and CD respectively. Taking $r_1=1$, arbitrarily, and $r_2=\sqrt{2}-1$, and solving the equations of the two parabolas simultaneously, we get

$$x = 2(2\sqrt{2} - 1)/7, \qquad y = (9 - 4\sqrt{2})/49,$$

whence

$$\sphericalangle PFN = \arctan y/(1 - x) = \arctan 1/7,$$

and

$$\sphericalangle MFG = \pi/4 - \arctan 1/7 = \arctan 3/4,$$

and the conclusion is established.

Also solved by Hüseyin Demir, E. I. Gale, A. R. Hyde, Beckham Martin, L. A. Ringenberg, David Rothman, C. W. Trigg, Roscoe Woods, and the proposer.

A Set of Finite Geometries

E 1098 [1954, 47]. *Proposed by L. A. Ringenberg, Eastern Illinois State College*

Construct all geometries satisfying the following axioms:

Axiom I. Space S is a set of n points, n a positive integer.

Axiom II. A line is a non-null subset of S.

Axiom III. Any two distinct lines have exactly one common point.

Axiom IV. Every point lies in exactly two distinct lines.

Solution by the Proposer. Since there is at least one point, there are at least two distinct lines (I, IV). Two such distinct lines, say a and b, have a point A in common; to be distinct, at least one of them, say a, contains a point other than A, say B. Then B must lie on some third line c (III, IV) distinct from a and b. It follows that the number m of lines in the geometry is an integer exceeding 2.

Two distinct pairs of distinct lines cannot have the same common point (IV). Hence $n=C(m,\ 2)=m(m-1)/2$. Thus a geometry with m lines has $n=m(m-1)/2$ points, each line consisting of $m-1$ points and each point lying on two lines.

Following is a diagram showing how to construct the geometry with $m=5$, $n=10$, the extension to an arbitrary $m\geqq 3$ is obvious.

line a: A B C D

b: A E F G

c: B E H I

d: C F H J

e: D G I J

Also solved by W. E. Briggs, Myles McConnon, D. C. B. Marsh, G. H. Meisters, J. D. Miller, D. B. Mumford, and D. R. Sudborough.

Editorial Note. The postulates form an interesting but simple set for classroom purposes. The student should examine the set for consistence, independence, categoricalness, and fertility.

A Gambling Table

E 1099 [1954, 47]. *Proposed by W. R. Van Voorhis, Fenn College*

On a certain gambling table there are N squares marked "2 to 1," "3 to 1," . . . , "$N+1$ to 1." A sum of money is placed on each square and one of the

squares is selected as a winner, paying the player at the odds marked on that square. The player loses the amounts placed on the other squares. What is the maximum N for which it is possible for a player to place money so that he can never suffer a net loss?

Solution by C. F. Pinzka, Princeton, N. J. Let S_k be the sum placed on the kth square and T the total amount bet. In order that there be no net loss, it is necessary and sufficient that $(k+2)S_k \geqq T$ for $k=1, 2, \cdots, N$. Writing this as $1/(k+2) \leqq S_k/T$, and summing both sides from $k=1$ to $k=N$, gives $\sum_{k=1}^{N} 1/(k+2) \leqq 1$. It is easily verified that 4 is the maximum N.

Also solved by W. E. Briggs, S. H. Eisman, A. L. Epstein, J. M. Howell, P. G. Kirmser, M. S. Klamkin, Bart Park, F. D. Parker, L. A. Ringenberg, D. C. Russell, N. Shklov, Hans Ury, and R. Z. Vause.

Ringenberg pointed out that not only can the player insure a net gain for $N=1, 2, 3, 4$, but that he can so bet that the net gain is independent of the square selected. If $N=2$, $S_1=4$, $S_2=3$, the gain is 5; if $N=3$, $S_1=20$, $S_2=15$, $S_3=12$, the gain is 13; if $N=4$, $S_1=20$, $S_2=15$, $S_3=12$, $S_4=10$, the gain is 3. Russell generalized the problem so that the stated odds are p_1 to 1, p_2 to 1, $\cdots$, p_n to 1.

An Inequality

E 1100 [1954, 47]. *Proposed by L. E. Ward, Sr., Naval Ordnance Test Station, China Lake, Calif., and L. E. Ward, Jr., University of Nevada*

For $x \geqq 0$ and $t \geqq 2$, prove that $(1+x^t)^{1/t} - (1+x^t)^{-1/t} \leqq x$.

Solution by J. V. Whittaker, U.C.L.A. Setting $1+x^t=y^t$, raising each side to the tth power, and dividing through by y^t, we obtain the equivalent inequality

$$(1 - 1/y^2)^t \leqq 1 - 1/y^t,$$

for $y \geqq 1$, $t \geqq 2$. But now

$$(1 - 1/y^2)^t \leqq (1 - 1/y^t)^t \leqq 1 - 1/y^t.$$

Equality is attained only when $y=1$. Hence in the given inequality, equality occurs only when $x=0$.

Also solved by Tom Black, W. E. Briggs, A. E. Danese, H. M. Feldman, Norman Greenspan, A. R. Hyde, M. S. Klamkin, Viktors Linis, A. E. Livingston, Lee Lorch, L. L .Pennisi, B. E. Rhoades, L. A. Ringenberg, R. Z. Vause, Chih-yi Wang, L. E. Ward, Jr., Albert Wilansky, R. E. Wild, and J. R. Winkelman.

Wilansky established the improved inequality

$$x - 1/x < L \leqq x^2(1 + x^2)^{-1/2},$$

where L is the original left hand side.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4598. *Proposed by I. J. Schoenberg, University of Pennsylvania*

Let the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \qquad (r < |z| < r^{-1}),$$

converge in an open ring containing the circle $|z| = 1$. Show that a sequence $\{b_n\}$ exists satisfying the conditions

(i) $|b_n| < K$, for some K, for all $n = 0, \pm 1, \pm 2, \cdots$,

(ii) $\sum_{n=-\infty}^{\infty} a_n b_{m-n} = 0$ for all m,

(iii) $b_n \neq 0$ for some n,

if and only if $f(z)$ vanishes somewhere on the circle $|z| = 1$.

4599. *Proposed by Leonard Carlitz, Duke University*

Let p be a prime >2, $a \not\equiv 0 \pmod{p}$. Show that the number of solutions of the congruence

$$(x + y + z)^2 \equiv 2axyz \pmod{p}$$

is p^2+1.

4600. *Proposed by Leon Bankoff, Los Angeles, California*

Vertices A–C and B–D of square $ABCD$ are joined by quadrants of circles (B) and (C). A semi-circle (O_1) is described internally on the diameter BC and a circle (O_2) is drawn tangent to the three arcs. Another circle (O_3) is drawn tangent to circle (O_2) and to arcs AC and BC, and a right triangle is formed by joining O_3 to O_1 and dropping a perpendicular from O_3 upon BC. Successively tangent circles are drawn in the same manner (with (O_n) tangent to (O_{n-1}) and to arcs AC and BC) and right triangles are formed (with O_1O_n for hypothenuse).

Show that the infinitude of triangles so constructed are Pythagorean.

4601. *Proposed by Harry Goheen, Iowa State College, Ames, Iowa*

What is the necessary and sufficient condition that as $i \to \infty$ the limit of n_i exists, if n_i is defined by the recurrence relation

$$n_{i+1} = bn_i^2 + c,$$

n_0 being given?

4602. *Proposed by C. M. Ablow and D. L. Johnson, Seattle, Washington*

Show that

$$\sum_{i=1}^{n} A_i \cos (B_i t + C_i)$$

changes sign as t varies from zero to positive infinity. The A_i, B_i, C_i are real constants; $A_i B_i \neq 0$.

SOLUTIONS

Squarefull Integers

4459 [1951, 636]. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Find an asymptotic expression for the number of integers, not exceeding x, each of which has the property that each of its prime divisors divides it to the second power at least.

Editorial Note. As has been pointed out by P. T. Bateman and others, the evaluation of the integral (1) of an earlier solution [1953, 719] is more than a direct application of Cauchy's theorem on residues and its complete justification requires tools far from simple.

Actually a rather good result can be obtained in this problem by a relatively elementary argument, as follows:

2. *Solution by P. T. Bateman, University of Illinois.* If c_n is 1 or 0 according as n does or does not have the property in question and if $A(x)$ is the number of integers not exceeding x having this property, then

$$A(x) = \sum_{n \leq x} c_n \tag{1}$$

and

$$\sum_{n=1}^{\infty} c_n n^{-s} = \zeta(2s)\zeta(3s)/\zeta(6s),$$

as in the former solution. Let b_n be the number of ways in which the positive integer n can be expressed in the form $m^2 n^3$, that is, as a product of a square and a cube. Thus

$$\sum_{j=1}^{\infty} b_j j^{-s} = \sum_{m=1}^{\infty} m^{-2s} \sum_{n=1}^{\infty} n^{-3s} = \zeta(2s)\zeta(3s)$$

for $R(s) > \frac{1}{2}$. We consider the summatory function

$$B(x) = \sum_{j \leqq x} b_j = \sum_{m^2 n^3 \leqq x} 1.$$

If m and n are any positive integers such that $m^2 n^3 \leqq x$, then either $m \leqq x^{1/5}$ or $n \leqq x^{1/5}$, possibly both. Thus

$$\begin{aligned} B(x) &= \sum_{m \leqq x^{1/5}} \sum_{n \leqq x^{1/3} m^{-2/3}} 1 + \sum_{m \leqq x^{1/5}} \sum_{m \leqq x^{1/2} n^{-3/2}} 1 - \sum_{m \leqq x^{1/5}} \sum_{n \leqq x^{1/5}} 1 \\ &= \sum_{m \leqq x^{1/5}} \left\{ x^{1/3} m^{-2/3} + O(1) \right\} + \sum_{n \leqq x^{1/5}} \left\{ x^{1/2} n^{-3/2} + O(1) \right\} - \left\{ x^{1/5} + O(1) \right\}^2 \\ &= x^{1/3} \sum_{m \leqq x^{1/5}} m^{-2/3} + x^{1/2} \sum_{n \leqq x^{1/5}} n^{-3/2} - x^{2/5} + O(x^{1/5}). \end{aligned}$$

Now we use the asymptotic formula

$$\sum_{n \leqq y} n^{-r} = y^{1-r}/(1-r) + \zeta(r) + O(y^{-r}),$$

where r is some fixed positive number other than 1, and $y \geqq 1$. (This formula is an easy consequence of the Euler-Maclaurin sum-formula of the first order; *cf.* Titchmarsh, *The Theory of the Riemann Zeta-Function*, Oxford, 1951, p. 14.) Using this in (2) we obtain

$$\begin{aligned} B(x) &= x^{1/3} \left\{ 3x^{1/15} + \zeta(2/3) + O(x^{-2/15}) \right\} \\ &\quad + x^{1/2} \left\{ \zeta(3/2) - 2x^{-1/10} + O(x^{-3/10}) \right\} - x^{2/5} + O(x^{1/5}) \\ &= \zeta(3/2) x^{1/2} + \zeta(2/3) x^{1/3} + O(x^{1/5}). \end{aligned} \tag{3}$$

Now if $R(s) > 1$, we have

$$1/\zeta(s) = \sum_{m=1}^{\infty} \mu(m) m^{-s},$$

where μ denotes the Möbius function. Accordingly

$$\sum_{n=1}^{\infty} c_n n^{-s} = \left\{ 1/\zeta(6s) \right\} \sum_{j=1}^{\infty} b_j j^{-s} = \sum_{m=1}^{\infty} \mu(m) m^{-6s} \sum_{i=1}^{\infty} b_i i^{-s}$$

(for $R(s) > \frac{1}{2}$) and thus

$$c_n = \sum_{m^6 j = n} \mu(m) b_j.$$

Hence by (3)

$$
\begin{aligned}
A(x) &= \sum_{m^6 j \leqq x} \mu(m) b_j = \sum_{m \leqq x^{1/6}} \mu(m) B(x/m^6) \\
&= \sum_{m \leqq x^{1/6}} \mu(m) \left\{ \zeta(3/2) x^{1/2} m^{-3} + \zeta(2/3) x^{1/3} m^{-2} + O(x^{1/6} m^{-6/5}) \right\} \\
&= \zeta(3/2) x^{1/2} \left\{ \sum_{m=1}^{\infty} \mu(m) m^{-3} + O\left(\sum_{m > x^{1/6}} m^{-3} \right) \right\} \\
&\quad + \zeta(2/3) x^{1/3} \left\{ \sum_{m=1}^{\infty} \mu(m) m^{-2} + O\left(\sum_{m > x^{1/6}} m^{-2} \right) \right\} \\
&\quad + O\left\{ x^{1/6} \sum_{m \leqq x^{1/6}} m^{-6/5} \right\}.
\end{aligned} \tag{4}
$$

Now if r is a fixed positive number greater than 1, and y is positive, $\sum_{n>y} n^{-r} = O(y^{1-r})$. Thus (4) gives

$$
\begin{aligned}
A(x) &= x^{1/2}\zeta(3/2)/\zeta(3) + O(x^{1/6}) + x^{1/3}\zeta(2/3)/\zeta(2) + O(x^{1/6}) + O(x^{1/5}) \\
&= x^{1/2}\zeta(3/2)/\zeta(3) + x^{1/3}\zeta(2/3)/\zeta(2) + O(x^{1/5}).
\end{aligned} \tag{5}
$$

This seems to be the best result readily obtainable by elementary methods. By more delicate methods, however, it is possible to sharpen the error term in (5) somewhat, *e.g.*, $O(x^{1/6} \log^2 x)$.

Inequality Involving Coefficients of Star-Like Functions

4535 [1953, 268]. *Proposed by M. S. Robertson, Rutgers University*

Let the function $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $a_1 \neq 0$, be regular for $|z| \leqq 1$ and let it map $|z| \leqq 1$ onto a simply-connected domain D, star-like when viewed from the origin. Show that

$$
2\left(\sum_{n=1}^{\infty} n |a_n|^2 \right)^2 \leqq \sum_{n=1}^{\infty} (n+1) |a_1 a_n + a_2 a_{n-1} + \cdots + a_n a_1|^2,
$$

with equality only for $f(z) = a_1 z$.

Solution by W. C. Royster, Alabama Polytechnic Institute. Let $f(z) = Re^{i\phi}$ and denote by C_1 the map of $|z| = 1$ effected by $w = f(z)$, and by C_2 the map of $|z| = 1$ effected by $w = [f(z)]^2$. Let A_1 be the area enclosed by C_1 and A_2 the area enclosed by C_2, the area on separate sheets counted separately. Then

$$
A_1 = \frac{1}{2} \int_{C_1} R^2 d\phi = \frac{1}{2} \int_0^{2\pi} R^2 \frac{\partial \phi}{\partial \theta} d\theta,
$$

$$
A_2 = \int_{C_2} R^4 d\phi = \int_0^{2\pi} R^4 \frac{\partial \phi}{\partial \theta} d\theta,
$$

where $z=re^{i\theta}$. Since $f(z)$ is starlike with respect to the origin, $\partial\phi/\partial\theta \geqq 0$ and $\int_0^{2\pi}(\partial\phi/\partial\theta)d\theta = 2\pi$. By the Schwartz integral inequality one has

$$\left(\int_0^{2\pi} R^2 \frac{\partial\phi}{\partial\theta}\,d\theta\right)^2 \leqq \int_0^{2\pi} R^4 \frac{\partial\phi}{\partial\theta}\,d\theta \int_0^{2\pi} \frac{\partial\phi}{\partial\theta}\,d\theta = 2\pi \int_0^{2\pi} R^4 \frac{\partial\phi}{\partial\theta}\,d\theta.$$

Thus $A_1^2 \leqq (\pi/2)A_2$, equality occurring only when R is constant. From this the required inequality follows since

$$A_1 = \sum_{n=1}^{\infty} n\,|\,a_n\,|^2, \qquad A_2 = \sum_{n=1}^{\infty} (n+1)\,|\,a_1a_n + \cdots + a_na_1\,|^2,$$

with equality holding only for $f(z)=a_1z$.

It is interesting to note that if $f(z)$ is p-valently starlike with respect to the origin then the right hand member of the given inequality is multiplied by p. This happens since $\int_0^{2\pi}(\partial\phi/\partial\theta)d\theta = 2\pi p$ for functions of this class. In this case equality holds only for $f(z)=a_pz^p$.

Also solved by the Proposer.

Uniform Convergence of Trigonometric Sums

4536 [1953, 268]. *Proposed by O. E. Stanaitis, St. Olaf College, Northfield, Minn.*

Prove that

$$\sum_{n=1}^{\infty} \frac{\sin n^2\theta \sin n\theta}{n}, \qquad \sum_{n=1}^{\infty} \frac{\cos n^2\theta \sin n\theta}{n}$$

are uniformly convergent in any interval, and that

$$\sum_{n=1}^{\infty} \frac{\sin n^2\theta \cos n\theta}{n}$$

is divergent.

Solution by E. M. Wright, University of Aberdeen, Scotland. The first series is uniformly convergent. We have

$$2\sum_{n=1}^{N} \frac{\sin n^2\theta \sin n\theta}{n} = \sum_{n=1}^{N} \frac{\cos n(n-1)\theta - \cos n(n+1)\theta}{n}$$

$$= 1 - \frac{\cos N(N+1)\theta}{N} - \sum_{n=1}^{N-1} \frac{\cos n(n+1)\theta}{n(n+1)}$$

which, as $N\to\infty$, becomes

$$1 - \sum_{n=1}^{\infty} \frac{\cos n(n+1)\theta}{n(n+1)}$$

uniformly in θ, and the last series is uniformly convergent. Similarly for the second given series, since

$$2 \cos n^2\theta \sin n\theta = \sin n(n-1)\theta - \sin n(n+1)\theta.$$

The third series converges for some values of θ and diverges for others. Let us put $\theta = p\pi/q$, where p, q are relatively prime positive integers and

$$S(N) = \sum_{n=1}^{N} \sin \frac{n^2 p\pi}{q} \cos \frac{np\pi}{q}.$$

Since $\sin (n^2 p\pi/q) \cos (np\pi/q)$ has period q, we have $S(rq) = rS(q)$ and

$$S(N) = \left[\frac{N}{q}\right] S(q) + S\left\{N - q\left[\frac{N}{q}\right]\right\} = \frac{N}{q} S(q) + f(N).$$

Thus $f(N)$, as just defined, satisfies $|f(N)| \leq 2q$. Hence, if $\theta = p\pi/q$,

$$\sum_{n=1}^{N} \frac{\sin n^2\theta \cos n\theta}{n} = \sum_{n=1}^{N} \frac{S(n) - S(n-1)}{n}$$

$$= \frac{S(q)}{q} \sum_{n=1}^{N} \frac{1}{n} + \sum_{n=1}^{N} \frac{f(n) - f(n-1)}{n}.$$

As $N \to \infty$ this last series is absolutely convergent by the Dirichlet test. Therefore the given series with $\theta = p\pi/q$ converges or diverges according as $S(q) = 0$ or $S(q) \neq 0$.

By combining the terms in $S(q)$ which correspond to $n = r$ and $n = q - r$, it is easy to see that $S(q) = 0$ for all even q (p is then odd). For $q = 3, 5$; $S(q) \neq 0$, evidently. It is somewhat more difficult to see that $S(q) \neq 0$ whenever q is odd; so that, in every interval, the given series converges for infinitely many values of θ and diverges for infinitely many others.

The problem is given in Bromwich, *Infinite Series*, 1926, p. 68, ex. 16, except for a misprint: the third series is printed whereas the second is intended.

Also solved by Joshua Barlaz, S. Parameswaran, and the Proposer.

Inversion of Order of Summation

4537 [1953, 268]. *Proposed by Albert Wilansky, Lehigh University*

Given that $\sum_n \sum_k a_{nk} x_k$ converges (as an iterated sum) whenever $\{x_k\}$ is a sequence such that $\sum |x_k| < \infty$. Show that $\sum_n \sum_k a_{nk} x_k = \sum_k \sum_n a_{nk} x_k$ for each such $\{x_k\}$.

Solution by R. P. Agnew, Cornell University. The required result and, moreover, necessary and sufficient conditions that a matrix a_{nk} have the property in question, follow immediately from theories of linear transformations originated by Silverman and Toeplitz in 1913 and greatly extended by Hans Hahn (Über Folgen linearer Operationen, *Monatshefte für Math. und Physik*, vol. 32 (1922), pp. 3–88) and others.

If we set

$$S_p = \sum_{n=1}^{p} \sum_{k=1}^{\infty} a_{nk} x_k, \qquad A_{pk} = \sum_{n=1}^{p} a_{nk} \tag{1}$$

then

$$S_p = \sum_{k=1}^{\infty} A_{pk} x_k, \tag{2}$$

and the hypothesis of the problem becomes the hypothesis that $S_1, S_2, \cdots$, and $\lim S_p$ exist whenever $\sum |x_k| < \infty$. Necessary and sufficient conditions for this are existence of constants $A_1, A_2, \cdots$, such that

$$\lim_{p \to \infty} A_{pk} = A_k \qquad k = 1, 2, \cdots \tag{3}$$

and existence of a constant M such that

$$|A_{pk}| \leqq M \qquad p, k = 1, 2, \cdots . \tag{4}$$

Moreover, when (3) and (4) hold and $\sum |x_k| < \infty$,

$$\lim_{p \to \infty} S_p = \sum_{k=1}^{\infty} A_k x_k. \tag{5}$$

This is the required result. In terms of the given matrix a_{nk}, the necessary and sufficient conditions (3) and (4) take the forms

$$\sum_{n=1}^{\infty} a_{nk} = A_k, \qquad \left| \sum_{n=1}^{p} a_{nk} \right| \leqq M. \tag{6}$$

The latter condition implies, but is not implied by, the condition

$$|a_{nk}| \leqq M', \qquad n, k = 1, 2, 3, \cdots . \tag{7}$$

An example, in which a_{nk} is the amusing matrix

$$\begin{array}{cccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & \cdots \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}$$

and $\sum |x_k|$ converges so slowly that $\sum k|x_k|$ diverges, shows that the series involved in the problem may fail to be absolutely convergent.

Also solved by V. F. Cowling, A. E. Livingston, and the Proposer, all of whom make explicit use of the properties of Banach spaces and refer to S. Banach, *Theorie des Operations Lineaires*, Warsaw (1932), p. 67.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

Calculus, A Modern Approach, Second Enlarged Edition. By Karl Menger, The Bookstore, Illinois Institute of Technology, 1953.

Of books on calculus there is no end. Yet seldom, if ever, does one of them make a new and significant contribution to the eternal problem of presenting the fundamental ideas correctly and systematically and "at a level of clarity and simplicity comprehensible to college students of mathematics, science and engineering." This book is exceptional; it is really new. And to some experienced teachers its newness may be disturbing, especially to those who until now have labored under the impression that existing textbooks are all right as they are.

Professor Menger explains in the preface the circumstances which prompted him to write the book: "Most mathematicians consider calculus as a domain about which they know everything. Twenty years ago the author, all in all, concurred in this view. Later, teaching the subject repeatedly and striving for greater clarity of the conceptual framework, he began to realize that he had underestimated the difficulties. In the late 1940's he ventured on a project to rescue calculus from one of the greatest dangers that may befall a discipline—the danger of petrification. A situation which rules out any essential progress can easily arise when specialists abstain from a critical study of books on a subject and at best skim the pages with an eye toward their possible use as texts for sophomore classes. In 1952, even after publishing the first edition of this book, the author felt that he should study calculus further."

The book is an outgrowth of a sequence of works by Professor Menger which have appeared since 1944, beginning with his *Algebra of Analysis*, and which have had as their aim a systematization and clarification of the foundations of analysis. These earlier works were not primarily intended for beginners. The present work is a textbook for beginners and for their instructors. In fact it is through its effect on this latter class that mathematical instruction will benefit most from the ideas which are here presented for the first time in a relatively simple manner and in textbook form.

It is not surprising that a reform of this kind should present striking changes. Among these the most striking at first are those of notation, and the author states in justification that " . . . the principal objective . . . is rather a clarification of the basic ideas and of their applications. In pursuing this aim the author found (to his regret) that changes in the notation were inevitable. In a certain sense these changes are only minor." It becomes clear after reading the book that the invention of the new notation was an essential step toward the clarification of the basic ideas and their applications and is thus amply justified. However, it is likely that these changes will be regarded by many readers as by no means minor ones.

There is so much in the book that is new that it is difficult to give a fair and comprehensive review of its content in a limited space. At the risk of injustice it may be said, briefly, that the main objectives of the book are the following:

(1) To clarify, in a manner which accords with modern points of view, the ideas underlying the basic terms: *number, class, function, variable quantity* (herein designated by V.Q.), *domain of a function, range of a function, domain of a V.Q., range of a V.Q., the derivative* (function) *of a function, anti-derivative* (function) *of a function, integral* (function) *of a function, inverse* (function) *of a function.*

(2) To study the operations performed on functions; the *algebraic* operations and that of *substitution*, and the basic operations of calculus—differentiation, integration—and their reciprocity laws.

(3) To present a new and adequate notation, free from ambiguities and inconsistencies, with which to represent the various concepts enumerated in (1) and (2).

(4) To provide an introduction to the application of these concepts to the physical sciences.

It goes without saying that all of the elementary functions are adequately presented.

Never before, in the reviewer's opinion, have these objectives been pursued in an elementary book with such painstaking care or with as much enthusiasm and vigor. As the author implies, this is no book to be skimmed over. Nearly every page must be read with scrutiny. The careful reader will detect several misprints not included in the list of errata, but these are almost all of the harmless variety or of the kind that will puzzle the reader to a point where he will be rewarded by a better understanding.

In Chapter I the author attacks the problems of differentiation and integration from the geometrical point of view. He defines a *simple curve* (graph) C as one which is intersected in at most one point by any line perpendicular to the X-axis. There is no Y-axis. The two problems are those of finding the slope of the curve C and the area under C. New notation is introduced at the outset:

$$\int_a^b C \quad \text{(read: the integral of } C \text{ from } a \text{ to } b) = \text{the area under } C \text{ from } a \text{ to } b.$$

$$\mathfrak{D}Cx \text{ (read: the derivative of } C \text{ at } x) = \text{the slope of } C \text{ at } x.$$

The horizontal line at the altitude c is denoted by $\underline{c}$. Thus

$$\int_a^b \underline{c} = c(b - a), \qquad \mathfrak{D}\underline{c}x = 0.$$

The letter I denotes the line which, for any point x on X, has the altitude x above the point x. Thus

$$Ix = x, \qquad \mathfrak{D}Ix = 1.$$

If L is any non-vertical line, there are two numbers c and c' such that the altitude of L above any point x on X is $xc+c'$. If this line is denoted by $cI+c'$ then

$$\mathfrak{D}(cI + c')x = \underline{c}$$

for every x on X.

If $\underline{c}$ is the slope line of L, then L is an area line of $\underline{c}$,

$$L \text{ is an } \int \underline{c} \text{ if and only if } \underline{c} = \mathfrak{D}L.$$

A line whose slope line or derivative is the horizontal line $\underline{c}$ is called an anti-derivative of $\underline{c}$ or $\mathfrak{D}^{-1}\underline{c}$.

$$L \text{ is a } \mathfrak{D}^{-1}\underline{c} \text{ if and only if } \mathfrak{D}L = \underline{c}.$$

Thus every $\int \underline{c}$ is a $\mathfrak{D}^{-1}\underline{c}$, and vice versa.

The word function is not discussed in Chapter I, but the notation used for a simple curve (graph) is the same as that used later to represent the corresponding function.

In Chapter II the study of the slope and area problems is continued by first introducing *step lines* and *polygonal lines*, the former being slope curves of the latter, and the latter being area curves of the former. Then the approximate area under a curve is found by drawing a step line, which approximates the given curve, and constructing the polygonal line which is the area curve of the step line. Similarly, the slope curve of the given curve is found approximately by means of an inscribed polygon whose slope curve gives approximately the slope of the given curve.

In Chapter III the student is shown how to perform the calculations corresponding to the approximate graphical solutions found in Chapter II. The formula giving the approximate solution of the area problem is expressed in the form

$$\int_a^b Q = c_1(x_1 - a) + c_2(x_2 - x_1) + \cdots + c_n(b - x_{n-1})$$

where Q is a step line with steps at heights c_1, $c_2 \cdots$. This leads to the fundamental inequalities for estimating the accuracy of the approximate solution, namely

$$\int_a^b \underline{Q}_n \leqq = \int_a^b C \leqq \int_a^b \overline{Q}_n$$

where $\underline{Q}_n$, $\overline{Q}_n$, are step lines whose steps are at heights equal to those of the low-

est and highest points of C, respectively, in the n intervals. These ideas are illustrated by copious exercises and examples.

Chapter IV is entitled *The Idea and the Use of Functions*. This chapter could be read with profit by almost every teacher of elementary calculus. In it the author, with admirable skill, leads the reader toward a thorough understanding of the function concept. He limits himself to functions whose domains are classes of numbers and whose ranges are also classes of numbers. Thus he states, among many examples, "If, with each number, 16 times the square of the number is paired (*e.g.* with 0 the number 0, with 1 the number 16, with 3/2 the number 36), then mathematicians say that a function has been defined. More precisely, the class of the pairs of numbers so defined is called a function. Traditionally, these definitions are expressed by saying that *with each number* x *the number* $16x^2$ *is paired*. The function is described as the class of the pairs $(x, 16x^2)$ for all numbers x."

"The pairs (0, 0), (1, 16), (1.5, 36), $(-1.5, 36)$, $(\sqrt{2}, 32)$ belong to the function, the pairs (16, 1), (3, 1/3), $\cdots$ do not, since $16 \cdot 16^2 \neq 1$, $16 \cdot 3^3 \neq 1/3$, $\cdots$. Clearly the same concept can be defined by saying that with each number t the number $16t^2$ is paired, or by considering the class of pairs $(a, 16a^2)$ for all numbers $a \cdots$. In each pair belonging to the function the *first* element is the number *with* which a number is paired. The class of all these first numbers is called the *domain* of the function. The *second* element in each pair is the number *which* is paired with the first. Each such number is called a *value* of the function .... The class of all values of a function is the *range* of the function ...."

"Clearly each function is a class of pairs of numbers. Two functions are *equal* if each pair of numbers belonging to the first function also belongs to the second, and each pair belonging to the second function also belongs to the first. Thus the function of which the pairs are $(x, 16x^2)$ for every x is equal to the function whose pairs are $(t, (4t-1)(4t+1)+1)$ for every t. ...."

"But not each class of pairs of numbers is a function. In order that a class of pairs of numbers be a function it is necessary and sufficient that in no two different pairs of the class are the first elements equal. ...."

"Can a function be equal to a number? Would anyone confuse a party of married couples with a spinster?"

In this long chapter the basic properties of the elementary functions are studied with unusual care: the constant functions, the functions I, I^n, polynomial functions, the exponential and logarithmic functions, trigonometric functions and their inverses. A great deal of space is devoted to the difficult general concept of the inverse, in which the author makes a clear distinction between pairs of functions f, f^* which are mutually inverse and those in which f^* is inverse to f but f is not inverse to f^*. This involves careful study, mainly geometrical, of the domains and ranges of functions, and is worth while. But the reader will be puzzled at first by the proposal, at the very end of the chapter, to define the class of elementary functions by means of the categories

I. (a) The constant functions $\underline{c}$;
(b) $\exp_{10}$, $\log_{10}$;
(c) arc $\tau = 2$ arc tan, $\tau = \tan(I/2)$;
(d) $I^{1/(2n+1)}$, $(n = 1, 2, 3, \cdots)$.

II. Functions which can be obtained from those in (I) by the fundamental operations: addition, multiplication, division, and substitution.

This is elegant but somewhat sophisticated for the beginner. He will have to read on in order to discover that I^c can be proved to be an elementary function in the domain $[>0]$, for every c, and will have to use this result to prove for himself that the function arc sin is elementary:

$$\text{arc sin} = \text{arc } \tau\left(\frac{I}{1+(1-I^2)^{1/2}}\right).$$

In fact the author gets the wrong answer.

Chapter V, *On Limits*, gives an easy introduction to the theory of limits. The author illustrates, in a common sense manner, all of the essential ideas without actually using deltas and epsilons. He explains precisely what is meant by writing for example

$$\text{if } y \underset{\text{suf}}{\sim} 2 \text{ and } z \underset{\text{suf}}{\sim} 6, \text{ then } y + z \underset{\text{arb}}{\sim} 8;$$

which is read: "if y is sufficiently close to 2 and z is sufficiently close to 6, then $y+z$ is arbitrarily close to 8." Using this notation he makes it easy for the student to understand what is meant by

$$\lim_{x\to 2}(x^2 - 1) = 3, \qquad \lim_{z\to 2}(z^2 - 1) = 3,$$

or, in his notation,

$$\lim_{2}(I^2 - 1) = 3,$$

and, in general,

$$\lim_{a} f = b.$$

The application of these ideas to the study of continuity is well done and the author takes the opportunity to identify "simple curves" with the graphs of continuous functions.

The author's intention here, as in other chapters, is to make the concepts clear and to state the facts correctly without supplying always the details of a complete proof. For example the number e is investigated by careful computations of the values of the expression $h = (1+x)^{1/x}$ for small values of x. From which, with the use of *tables of logarithms*, it is then made clear that $\log_{10} e$ lies between .434 and .435.

Chapter VI, *The Basic Concepts of Calculus*, deals with derivatives, antiderivatives, and integrals, by utilizing the analysis developed in Chapter V. Here the fundamental theorems of integral calculus are explained and made plausible. Applications involving elementary functions are plentiful and the exercises are good. The chapter ends with a fairly thorough discussion of the Reciprocity Laws:

$$\text{I.}\quad \mathfrak{D}\int_a f = f \qquad (f \text{ continuous}),$$

$$\text{II.}\quad \int_a \mathfrak{D}f = f - f_a.$$

Law I is proved. Law II is not proved at this point because the Theorem of the Mean, needed in the proof, does not appear until Chapter X. The reviewer believes that the postponement of this basic theorem is unwise as it is essential to the study of the increase and decrease of functions and of problems in maxima and minima, a subject which has important applications, and which, on account of its appeal to the student, should come as early as possible in the course. It is noticeable that the author introduces Law II with a misleading statement to the effect that it will be proved in a later chapter *for any function f having a derivative* $\mathfrak{D}f$. Of course the proof, in Chapter X, is based on the continuity of $\mathfrak{D}f$.

In Chapter VII, *Applications to Physical Science*, the author treats the difficult concepts of mensuration, variable quantity, and function, as applied to physical sciences. It is not easy to give an adequate idea of the effectiveness with which these concepts are developed. By easy stages, starting with a class Σ of elements σ, σ', $\cdots$ of undetermined nature, which can be *compared with each other* by means of the usual order and equivalence relationships $\sigma \preceq \sigma'$, $\sigma \sim \sigma'$, there is established a pairing of the elements of Σ with numbers of a numerical scale so that with each element σ of Σ a number $n\sigma$ is paired—just as in the definition of function—but such that if $\sigma \preceq \sigma'$ then $n\sigma \leqq n\sigma'$, and $n\sigma = n\sigma'$ if and only if $\sigma \sim \sigma'$. By postulating the additive property for the elements of Σ and the existence of a unit element σ_1 (an element with which the number 1 is paired), there is established a *mensuration scale*. Thus, if $\sigma \sim \sigma_1 + \sigma_1 + \sigma_1$, then with σ is paired the number 3 and one writes $\sigma = 3\sigma_1$; and if $\sigma + \sigma + \sigma = \sigma_1 + \sigma_1$, then with σ the number $\frac{2}{3}$ is paired and one writes $\sigma = \frac{2}{3}\sigma_1$. And by an obvious extension the statement $\sigma = \sqrt{2}\sigma_1$, is defined. The ideas are copiously illustrated of course. The notion of the unit element is illustrated, for example, by such common statements of mensuration as:

"The rod R is 7.5 ft.," "the weight P is $\frac{1}{2}$ lb.,"

which yield the corresponding equivalent formulae:

(a) $$R = 7.5 \text{ ft.}, \qquad P = \tfrac{1}{2} \text{ lb.}$$

Each such formula is an equality of an observable and a so-called denominate

number (a pure number followed by an observable comparable with the first). Menger then defines a preferred notation, which consists in replacing $\sigma' = k\sigma$ by $(\sigma' \text{ in } \sigma) = k$. So that, in the example, (a) above is replaced by:

(b) $$(R \text{ in ft.}) = 7.5, \qquad (P \text{ in lbs.}) = \tfrac{1}{2}.$$

By easy stages we then arrive at the *Fundamental Law of Mensuration*:

$$\begin{aligned} &\textit{If} && (\sigma' \text{ in } \sigma) = a \\ &\textit{and if} && (\sigma'' \text{ in } \sigma') = b, \\ &\textit{then} && (\sigma'' \text{ in } \sigma) = ab. \end{aligned}$$

The author is now prepared to answer the question "What is a variable as studied in Science?" The answer is:

"If a number is paired with each element of a class Σ—the same number with any two equivalent elements—then we call the class of pairs (element, number paired with the element) a *variable*, more specifically, a variable with the domain Σ. For example, if Σ is additive and a unit σ_1 has been selected in Σ, the class of pairs

$$(\sigma, \sigma \text{ in } \sigma_1) \qquad \text{for all elements } \sigma \text{ of } \Sigma$$

is a variable with the domain Σ. More generally, every numerical scale in a class Σ of comparable elements gives rise to a variable, namely the class of pairs $(\sigma, n\sigma)$, for all σ, where $n\sigma$ is the label of σ in the numerical scale. The numbers paired with the elements are called the *values* of the variable, and the class of all values of the variable its *range*."

Thus, for example, if in the class of rods having length, a foot is selected as the unit, for each rod ρ we have the pairing, $(\rho, \rho \text{ in ft.})$, and the class of all such pairings, for all rods ρ, is an *observable variable*, called the length of rods. The variable is denoted by the letter l. Its values are $l\rho = (\rho \text{ in ft.})$.

Functions are the only variables whose domains are numbers. From this it follows that into a function a variable can be substituted provided that the range of the variable is a part of the domain of the function.

The chapter culminates in a penetrating exposition of the central question: When is a variable m a function of another variable n?

The fundamental idea comes from the definition of functionally related classes: Let Σ and T be two classes. A class of pairs (τ, σ) such that τ belongs to T and σ belongs to Σ is called a *pairing of* Σ *with* T. Σ is *functionally related to* T *with regard to the pairing* Π if any two elements of Σ which are paired with two equivalent elements of T are equivalent; in other words, if whenever (τ, σ) and (τ', σ') belong to Π and $\tau \sim \tau'$ then $\sigma \sim \sigma'$.

This gives the central theorem: If the class Σ is functionaliy related to the class T with regard to the pairing Π, and numerical scales are established, pairing labels $m\sigma$ and $n\tau$ with all elements of Σ and T respectively, then the variable m is a function of the variable n with regard to the pairing Π; that is to say,

$m=fn$ for some function f. [There is an unfortunate misprint in the text, not noted in the *Errata*; the result is given as $n=fm$.]

The rest of this long chapter is devoted to the study of derivatives and integrals of functions connecting variable quantities. To illustrate: Let s be the distance travelled by a particle (in units of length), t the time elapsed since the beginning of the observation of the motion (in units of time), s_t the function connecting s with t. If τ, τ', are the instants of time at which t has the values x, x', respectively, σ, σ', the positions corresponding to those instants of time, so that

$$s\sigma = s_t\tau, \qquad s\sigma' = s_t\tau',$$

then the difference quotient

$$\frac{s\sigma' - s\sigma}{t\tau' - t\tau} = \frac{s_t\tau' - s_t\tau}{t\tau' - t\tau} = \frac{s_tx' - s_tx}{x' - x}$$

is the average speed of the particle between x and x' (that is, between the instants τ and τ'). If this quotient has a limit as $x' \to x$ then this limit, which is $\mathcal{D}s_tx$, is called the *instantaneous speed* of the particle *at* x (that is, at the instant τ). In symbols,

$$\mathcal{D}s_tx = \lim_{x' \to x} \frac{s\sigma' - s\sigma}{t\tau' - t\tau} = \lim_{x' \to x} \frac{s_tx' - s_tx}{x' - x}$$

is the instantaneous speed at x. Using v to represent the variable quantity *instantaneous speed of the particle*, then

$$v_tx = \mathcal{D}s_tx, \qquad \text{for every number } x,$$

or

$$v_t = \mathcal{D}s_t.$$

Another way to describe this relation is to say that:

If $s_t = f$, then $v_t = \mathcal{D}f$. For example;

If $s_t = 16t^2$ then $v_t = 32t$.

These are functions. The traditional way of expressing these ideas would be: If f is the function connecting s with t then

$$s = ft, \qquad v = \mathcal{D}ft.$$

These quantities are variables. In particular:

$$\text{if } s = 16t^2 \text{ then } v = 32t.$$

To illustrate the use of integrals the author considers, among other examples, the work done by a force p acting on a particle (the rectilinear case). If p_s is the function connecting p with the distance travelled s, we have

$$ {}_aw_s = \int_a p_s. $$

This is *the work function from a on*. The *variable* ${}_aw$ *is the work from a on*.

$$ {}_aw = \left(\int_a p_s \right) s. $$

For example, if $p_s = -I$ then $\int_a(-I) = -\frac{1}{2}I^2 + \frac{1}{2}a^2$ and

$$ {}_aw = -\frac{1}{2}s^2 + \frac{1}{2}a^2. $$

More generally, if u and v are two variables, v a function of u, then ${}_aw_u = \int_a v_u$ is called the *cumulation function of v with u from a on*. The variable ${}_aw$, *the cumulation* of v with *u from a on*, is connected with u by the cumulation function as follows:

$$ {}_aw = \left(\int_a v_u \right) u. $$

The rest of the book need not be discussed in detail. In fact the most important parts are those, already discussed, in which the fresh point of view of the author is set forth. In the remaining chapters the ideas presented in the first seven chapters are applied to the techniques of calculus which, in turn, are applied to problems in analytic geometry and mechanics.

Chapter VIII, *The Calculus of Derivatives*, gives a thorough exposition of the techniques used in calculating the derivatives of functions expressed in terms of differentiable functions by means of the fundamental operations (addition, multiplication, taking the reciprocal, and substitution).

The operation of differentiation is represented by the symbol $\mathcal{D}$ when applied to a pure function. $\mathcal{D}f+g$ means $(\mathcal{D}f)+g$, $\mathcal{D}f \cdot g$ means $(\mathcal{D}f) \cdot g$, $\mathcal{D}fg$ means $(\mathcal{D}f)g$; the derivatives of $f+g, f \cdot g, fg$ are denoted by $\mathcal{D}(f+g), \mathcal{D}(f \cdot g), \mathcal{D}(fg)$.

One of the main features of this chapter is the illuminating discussion of the meaning of the various notations used in representing the derivative of a function and the derivative of one variable with respect to another (with which the first is functionally related).

Chapter IX, *The Calculus of Antiderivatives*, is a lucid exposition of methods of finding elementary antiderivatives. It affords thorough training in the basic techniques and in the use of the Menger notation.

Chapter X, *The Mean Value Theorem and Its Consequences*, begins with a common sense treatment of the approximate value of a function near a point a by means of its value at a and the value of the derivative at a. This is followed by a Mean Value Theorem, which is proved by means of Rolle's Theorem which, in turn, is rendered plausible by a clear geometrical argument. The chapter culminates in Taylor's Theorem with the Lagrange form of the remainder. The rest

of the chapter deals with the usual applications to computation with due attention to estimations of the remainder term.

Chapter XI, *Two-Place Functions or Functions of Two Variables*. This final chapter introduces new notation for the study of functions of several variables. Capital letters are used to denote two-place functions and all one-place functions are denoted by lower case letters. Here I is the two-place function such that $I(x, y) = x$ for every (x, y), J is the two-place function such that $J(x, y) = y$ for every (x, y). The one-place function denoted in the preceding chapters by I, such that $It = t$ for every t is now denoted by ι. Graphs (surfaces) of simple two-place functions are studied. The idea of substituting one-place and two-place functions in a two-place function is given careful treatment. This idea is applied to the differentiation of one-place functions defined implicitly. Partial differentiation of functions is expertly presented but only the simpler techniques are treated in the text proper. The differentiation of a one-place function obtained by substituting two one-place functions in a one-place function is the only case of the Substitution Rule considered. There is an interesting but brief Appendix devoted to the study of partial differentiation of n-place functions and a statement of the general Substitution Rule.

There is little doubt that this book deserves the most careful attention of teachers of calculus. It is written in the lively, vigorous style which is characteristic of the author and on this account it should appeal to any earnest student. Opinions among teachers may vary widely as to the seriousness of the defects of the traditional point of view of which the author complains. But there can be little difference of opinion as to the effectiveness of the remedies which he supplies. The fact is that even an experienced teacher who reads the book with care will probably find that he has acquired a deeper insight into the significance of the fundamental ideas and a better understanding of the art of presenting them to beginners. It must not be dismissed with the pointless remark that it is a book on function-theory, not calculus. The two are inseparable. But there is a debatable question: How deeply should one go into the foundations of analysis in writing a book for beginners? Some teachers prefer a book that skims over the surface of the subject because it affords them free opportunity to present the deeper ideas in the class room according to their own tastes. This is not that kind of book. But it still leaves opportunities for the teacher to introduce supplementary material, since much that is traditionally included in a calculus book is omitted; for example, drill exercises and problems of a so-called practical nature drawn from geometry, mechanics, *etc.*

Unfortunately the typography of the book is unattractive, but this fault will undoubtedly be remedied by an enterprising publisher who will print the book in the attractive format in which so many lesser books are now being offered.

H. E. Bray
The Rice Institute

NEW BOOKS RECEIVED

History of the Theories of Aether and Electricity. By Sir Edmund Whittaker. New York, Philosophical Library, 1954. xi+319 pages. $8.75.

College Algebra. New Third Edition. By P. K. Rees and F. W. Sparks. New York, McGraw-Hill Book Company, Inc., 1954. xiii+460 pages. $4.25.

Analytic Geometry. Second Edition. By E. S. Smith, Meyer Salkover and H. K. Justice. New York, John Wiley and Sons, Inc., 1954. xiii+306 pages. $4.00.

Understanding College Algebra. By E. R. Smith, Samuel Selby and Murray Kleiman. New York, The Dryden Press, 1954. xvi+571 pages. $3.50.

Elements of Statistics. By H. C. Fryer. New York, John Wiley and Sons, Inc., 1954. vi+262 pages. $4.75.

An Introduction to the Calculus of Finite Differences. By C. H. Richardson. New York, D. Van Nostrand Company, Inc., 1954. v+142 pages.

Theory of Equations. By C. C. MacDuffee. New York, John Wiley and Sons, Inc., 1954. vii+120 pages. $3.75.

Tables of Integral Transforms. By A. Erdélyi. New York, McGraw-Hill Book Company, Inc., 1954. xx+391 pages. $7.50.

Matter Energy Mechanics. By Jakob Mandelker. New York, Philosophical Library, 1954. ix+73 pages. $3.75.

First Course in Calculus. By H. R. Cooley. New York, John Wiley and Sons. Inc., 1954. xii+643 pages. $6.00.

Introductory College Mathematics. By Adele Leonhardy. New York, John Wiley and Sons, Inc., 1954. ix+459 pages. $4.90.

Introductory College Mathematics. By C. G. Jaeger and M. M. Bacon. New York, Harper and Bros., 1954. xii+382 pages. $4.75.

The Mechanism of Economic Systems. By Arnold Tustin. Cambridge, Harvard University Press, 1954. xi+161 pages. $5.00.

Number. Fourth Edition. By Tobias Dantzig. New York, The Macmillan Company, 1954. ix+340 pages. $5.00.

Grundzuge der Ausgleichungsrechnung. By Walter Grossmann. Berlin, Germany, Springer-Verlag, 1953. 261 pages. $4.56.

The Japanese Abacus, its Use and Theory. By Takashi Kojima. Rutland, Vermont, Charles E. Tuttle Company, 1954. 102 pages. $1.25.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

SALE OF MATHEMATICAL LIBRARY

The library of the late Professor Otto Szász is available for sale. It contains over 100 mathematical books of importance, complete sequences of the *Bulletin of the American Mathematical Society*, the *Journal of the London Mathematical Society*, and the *Proceedings of the London Mathematical Society* from 1933–1952, and other useful periodicals. To be mentioned also are the Proceedings of various mathematical congresses from 1912 on. In addition, about 4500 reprints are available. Interested persons may contact Mrs. Brigitta Dobratz, 2911 Knoll Drive, Concord, California.

PERSONAL ITEMS

The following have received Guggenheim Fellowships for 1954–55 for the studies indicated:

Professor C. B. Boyer, Brooklyn College, studies in the history of the theory of the rainbow;

Associate Professor D. G. Chapman, University of Washington, studies of the size and dynamic structure of wild animal populations;

Professor S. S. Chern, University of Chicago, studies in the field of differential geometry;

Professor M. R. Hestenes, University of California, Los Angeles, study of the applications of the theory of linear spaces to the calculus of variations;

Associate Professor E. R. Kolchin, Columbia University, study of certain questions in the Galois theory of differential fields;

Professor C. C. Lin, Massachusetts Institute of Technology, studies in the field of theoretical fluid dynamics;

Assistant Professor F. I. Mautner, Johns Hopkins University, research on unitary representations of locally compact topological groups;

Professor Hans Rademacher, University of Pennsylvania, studies of generalized modular functions belonging to real quadratic fields;

Assistant Professor M. A. Rosenlicht, Northwestern University, study of non-complete group varieties;

Research Professor A. H. Taub, University of Illinois, studies of the gravitational field in relativity and of curved shocks in hydrodynamics;

Professor Alexander Weinstein, University of Maryland, study of initial and boundary value problems for singular hyperbolic equations and for partial differential equations of mixed type.

National Science Foundation Postdoctoral Fellowships have been awarded to the following: G. E. Baxter, University of Minnesota; R. C. Blanchfield,

Princeton University; D. A. Buchsbaum, Columbia University; C. W. Curtis, Yale University; Jacob Feldman, Columbia University; W. M. Huebsch, University of Notre Dame; A. P. Mattuck, Princeton University; J. C. Moore, Princeton University; Edgar Reich, University of California; M. F. Ruchte, University of Wisconsin; V. L. Shapiro, Institute for Advanced Study; Edwin Weiss, Massachusetts Institute of Technology.

Professors B. W. Jones of the University of Colorado and Einar Hille of Yale University represented the Mathematical Association of America at the International Congress of Mathematicians which was held in Amsterdam, The Netherlands, September 2–9, 1954.

Assistant Professor A. N. Aheart of West Virginia State College represented the Association at the inauguration of President W. J. L. Wallace of the College on April 9, 1954.

Professor H. J. Ettlinger of the University of Texas was the representative of the Association at the inauguration of President D. H. Morgan of Texas Agricultural and Mechanical College on May 20, 1954.

Professor Fritz John of the Institute for Mathematics and Mechanics, New York University, was appointed to represent the Association at the Second University Convocation of the Bicentennial Year of Columbia University which was held on June 1, 1954.

Brown University announces the following: Dr. John Wermer has been appointed to an assistant professorship; Assistant Professor Joanne Elliott of Mt. Holyoke College has been appointed Visiting Assistant Professor for the year 1954–55; Professor R. E. Gilman has retired with the title of Professor Emeritus. The University reports also that a program of instruction leading to the Bachelor of Science in Applied Mathematics will be offered beginning Fall, 1954; Professor E. H. Lee, chairman of the Division of Applied Mathematics, is in charge of the program.

At Montana State University: Dr. A. S. Merrill, dean of the faculty and professor of mathematics, has been appointed Vice-President of the University while retaining his other titles; Professor Harold Chatland, chairman of the Department of Mathematics, has been appointed Dean of the College of Arts and Sciences.

Randolph-Macon Woman's College announces: Professor Evelyn P. Wiggin is on leave during the academic year 1954–55 and is studying at the University of Chicago; Miss Sara L. Ripy has been appointed to an instructorship.

Rutgers University reports the following: Assistant Professor R. M. Cohn has been promoted to an associate professorship; Dr. B. H. McCandless, Dr. Solomon Leader, Dr. V. L. Shapiro, and Dr. K. G. Wolfson have been promoted to assistant professorships; Mr. N. R. Stanley is transferring from the New Brunswick Branch to the Newark Colleges of the University; Mr. A. G. Wootton has resigned.

University of Connecticut announces the following appointments: Dr. Allen Devinatz of the Institute for Advanced Study to an assistant professorship; Dr.

R. L. Ingraham of the Institute for Advanced Study and Dr. H. C. Griffith of the University of Tennessee to instructorships.

University of Illinois announces: Professor D. G. Bourgin has a Fulbright Grant to Rome for the academic year 1954–55; Professor S. S. Cairns has a Fulbright Grant to the University of Strasbourg in France and a sabbatical leave for the year 1954–55; Assistant Professor Irving Reiner is on sabbatical leave during 1954–55; Assistant Professor Lowell Schoenfeld is on sabbatical leave during 1954–55 and is dividing his time between the University of Pennsylvania and the Institute for Advanced Study; Dr. Alex Heller of Harvard University has been appointed to an assistant professorship.

At the University of Wisconsin: Associate Professor R. C. Buck has been promoted to a professorship; Assistant Professor W. F. Eberlein has been promoted to an associate professorship; Dr. R. C. T. Smith of New England University College, New South Wales, has been appointed Visiting Lecturer for the academic year 1954–55; Assistant Professor Elizabeth Sokolnikoff Hirschfelder has resigned.

University of Wichita reports the following: Assistant Professors Harold Huneke and Ferna Wrestler have been promoted to associate professorships; Miss Jeneva J. Brewer has been promoted to an assistant professorship; Miss Ann Klein has been appointed to an instructorship.

Wayne University announces the following appointments to instructorships: Mr. W. S. Bicknell, Mr. Charles Briggs, Dr. Shu-Teh C. Moy, and Dr. E. S. Northam.

Associate Professor B. H. Bissinger of Lebanon Valley College has been elected Chairman of the Department of Mathematics.

Mrs. Elizabeth B. Bockelman, formerly with the Acoustics Research Laboratory of Harvard University, has been appointed to an instructorship at Wellesley College.

Professor H. K. Brown of Northeastern University has been promoted to the position of Dean of the Graduate Division.

Associate Professor D. E. Christie has returned to Bowdoin College after a year's leave of absence as a visiting Fellow at Princeton University.

Mr. Albert Derin of Kansas State College is now in Military Service.

Assistant Professor D. G. Duncan of the University of Arizona has accepted an appointment at San Jose College.

Mr. J. J. Gehrig of the Ballistics Research Laboratories, Aberdeen Proving Ground, has accepted a position as research engineer with Northrop Aircraft, Incorporated, Hawthorne, California.

Mr. M. N. Haller, Jr., previously an engineer with Southern Bell Telephone and Telegraph Company, is now engaged as an engineer by Capehart-Farnsworth Company, Fort Wayne, Indiana.

Mr. P. W. M. John, who has been teaching at the Casady School, Oklahoma City, has been appointed to an instructorship at the University of Oklahoma.

Mr. H. T. LaBorde of the University of North Carolina has been appointed

to an assistant professorship at the University of the South.

Dr. H. N. Laden has accepted a position as senior methods research officer with the Chesapeake and Ohio Railway Company, Cleveland, Ohio.

Mr. P. M. Landry, formerly with the Vitro Corporation, Fort Walton, Florida, is now in Military Service.

Dr. H. M. MacNeille, who has resigned as Executive Director of the American Mathematical Society effective September 1, 1954, has been appointed Professor and Chairman of the Department of Mathematics of Washington University.

Associate Professor R. R. Otter has returned to the University of Notre Dame after a year as Visiting Professor at Princeton University.

Assistant Professor L. J. Paige, who spent the year 1953–54 at the Institute for Advanced Study, has returned to the University of California.

Associate Professor T. S. Peterson, Portland State Extension Center, has been promoted to a professorship.

Mr. S. L. Pollack of the National Bureau of Standards has accepted a position as mathematician with the Raytheon Manufacturing Company, Waltham, Massachusetts.

Dr. Edgar Reich of the Rand Corporation is spending the academic year 1954–55 at the Institute for Advanced Study as a National Science Foundation fellow.

Mr. Yomei Sawanobori, formerly at the Cornell Aeronautical Laboratory, Buffalo, New York, has a position as a mathematician with the I. B. M. Corporation, New York City.

Dr. Alice T. Schafer has been appointed to an assistant professorship at Connecticut College.

Assistant Professor M. R. Spiegel of Rensselaer Polytechnic Institute has been promoted to an associate professorship.

Dean G. W. Starcher of the College of Arts and Sciences of Ohio University has been appointed President of the University of North Dakota.

Assistant Professor B. B. Townsend of Louisiana State University has been promoted to an associate professorship.

Professor J. F. Wampler, previously professor of mathematics and registrar at York College, has been appointed to an associate professorship at Nebraska Wesleyan University.

Dr. R. C. Briant, project director at Oak Ridge National Laboratory, died on April 25, 1954. He had been a member of the Association for twenty-seven years.

Professor Emeritus H. S. Brown of Hamilton College died on March 27, 1954. He was a charter member of the Association.

Professor J. A. Daum of Texas Agricultural and Mechanical College died on March 23, 1954.

Professor Emeritus Arnold Dresden of Swarthmore College died on April 10,

1954. He had served as President of the Association during 1933–34, as Vice-President during 1931, and as a member of the Board of Governors for the periods 1935–40 and 1943–45.

Professor Emeritus William Findlay of McMaster University died on May 7, 1953. He was a charter member of the Association.

Professor D. L. Holl, head of the Department of Mathematics of Iowa State College, died on May 22, 1954. He was a member of the Iowa Section for nine years. He served as Section Chairman in 1950–51 and had been elected Governor of the Iowa Section for the period 1953–56.

Professor E. L. Post of City College of the City of New York died on April 21, 1954. He had been a member of the Association for thirty-one years.

Mr. J. E. Sanders, retired meteorologist of Knoxville, Tennessee, died on April 8, 1953. He was a charter member of the Association.

Professor J. M. Stetson, William and Mary College, who was a charter member of the Association, died on May 5, 1954.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1954 by a mail vote of the membership of the Association in the Sections indicated:

Allegheny Mountain	Morris Ostrofsky, Westinghouse Electric Corp.
Indiana	Ralph Hull, Purdue University
Kentucky	H. H. Downing, University of Kentucky
Metropolitan New York	R. M. Foster, Polytechnic Institute of Brooklyn
Nebraska	M. A. Basoco, University of Nebraska
Northern California	W. H. Myers, San Jose State College
Oklahoma	L. W. Johnson, Oklahoma A. and M. College
Rocky Mountain	C. A. Hutchinson, University of Colorado
Wisconsin	R. C. Huffer, Beloit College

Over 50% of the members eligible to vote actually cast their votes in the Kentucky Section and the Wisconsin Section. In general, a high percentage of votes is cast in the elections of Sectional Governors, thus indicating the desire of members of the Association to have their Sections represented by outstanding candidates.

H. M. GEHMAN, *Secretary-Treasurer*

THE 1954 COMBINED MEMBERSHIP LIST

Each year the Mathematical Association of America and the American Mathematical Society issue a Combined Membership List. The 1954 edition of this List will be sent to members of the Society.

The office of the Association should be notified promptly of all changes in rank, position, and address which have not previously been reported. The final date for receipt of changes is October 15. Any errors in the 1953 Membership List should also be reported before October 15.

H. M. GEHMAN, *Secretary-Treasurer*

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 89 persons have been elected to membership by the Board of Governors on applications duly certified.

S. S. ABHYANKAR, A.M. (Harvard) Teaching Fellow, Harvard University.

A. G. ASHOOK, A.B. (Boston C.) Teacher, Oliver W. Holmes Jr. High School, Dorchester, Massachusetts.

SHELDON BALK, Student, Arizona State College, Tempe.

G. E. BARLOW, JR., B.S. (V.P.I.) Assistant County Agent, Gate City, Virginia.

G. C. BARTON, B.A. (U. of Washington) Teacher, Concrete Senior High School, Washington.

R. S. BEARD, M.S. (Kansas) Col. U.S.A. retired; Part-time Prof., Yeshiva Institute of Mathematics.

C. P. BENNER, M.S. (C.I.T.; Houston) Asst. Prof., University of Houston.

H. F. BENNETT, B.S. (Illinois Wesleyan) Patent Agent, Eastman Kodak, Rochester, N. Y.

RENE BLOCH, Student, University of Basel, Switzerland.

J. D. BUCKHOLTZ, Student, Little Rock Junior College.

REV. FR. FLORENT CARTUYVELS, Lic. in Math. (Louvain) Teacher, St. Xavier's College, Ranchi, India.

C. P. CEBULLA, M.Ed. (St. Thomas) Instr., Hermantown High School, Duluth, Minn.

J. R. COX, Student, Lebanon Valley College.

J. B. CRAWFORD, B.S. (U. S. Military Academy) Asst. Prof., Alabama Polytechnic Institute.

C. L. DAVIS, M.A. (Michigan) Instr., General Motors Institute, Flint, Michigan.

D. E. DEAN, B.S. (Akron) Technical Supervisor, Firestone Tire & Rubber, Akron, Ohio.

T. P. DENNEHY, M.S. (Notre Dame) Instr., John Carroll University.

R. S. DICK, Student, Queens College.

OLIVE A. DOUGHTY, B.S. (Northeast La. S.C.) Mathematician, David Taylor Model Basin.

B. M. DRUCKER, Ph.D. (North Carolina) Asst. Prof., Georgia Institute of Technology.

G. E. DUNCAN, Student, Georgia Institute of Technology.

G. A. EASTERLY, B.S. (Allen) Teacher, Morristown College High School, Tennessee.

E. J. FARRELL, M.A. (Stanford) Asst. Prof., University of San Francisco.

M. A. FELDSTEIN, Student, Arizona State College, Tempe.

C. T. FIKE, Student, University of the South.

L. R. FORD, JR., Ph.D. (Illinois) Research Instructor, Duke University.

L. R. FOY, M.S. (St. John's) Instr., Marian College, Poughkeepsie, N. Y.

J. D. GATES, A.B. (Hiram) Teacher, Maple Heights High School, Ohio.

J. B. GOEBEL, Student, University of Oregon.

S. W. GOLOMB, M.A. (Harvard) Graduate Student, Harvard University; Instr., Boston University.

T. E. HAGENSEE, M.S. (DePaul) Teacher, Crane Technical High School, Chicago, Ill.

D. I. HAMMER, M.A. (Columbia) Instr., Montclair State Teachers College.

W. G. HANKS, Student, University of Manitoba.

M. E. HARRIS, Student, Yale University.

H. L. HENDRICKS, B.A. (Pomona) Teacher, Pasadena City College.

PFC. R. A. HULTQUIST, M.S. (Purdue) Research Assistant, Fort Sill, Oklahoma.

REV. L. E. ISENECKER, S.J., Ph.B. (Loyola) Graduate Student, Catholic University.

DIANE M. JOHNSON, Student, University of Manitoba.

VIRGINIA L. JOHNSON, Student, Harpur College.

MRS. CATHERINE S. KAY, M.A. (Michigan) Instr., North Central College, Naperville, Illinois.

C. S. KIMBALL, Tech. Representative, Philco Corporation, Philadelphia, Pa.

J. E. KIST, M.S. (Purdue) Research Assistant, Purdue University.

S. M. KLEINMAN, Brooklyn, New York.

DIANE L. KOCHER, Student, Harpur College.

A. R. LAMONTAGNE, M.A. (New Hampshire) Graduate Assistant, University of New Hampshire.

ADELE LEONHARDY, Ed.D. (Missouri) Chairman of Dept., Stephens College.

K. R. LUCAS, B.S. (Washburn) Graduate Student, University of Kansas.

DALE MANESS, M.A. (Vanderbilt) Asst. Prof., Baylor University.

M. D. MARCUS, A.B. (California) Research Assistant, University of California at Berkeley.

H. J. MCBLAINE, JR., M.S. (Illinois) Training Instr., Meteorology, Dept. of Weather School, Chanute AFB, Ill.

WILLIAM MCKAY, B.S. (Drexel) Instr., Drexel Institute of Technology.

D. E. MCOWEN, M.A. (Michigan) Instr., General Motors Institute, Flint, Michigan.

B. T. MENDELSOHN, Student, City College of New York.

A. A. MULLIN, Student, Syracuse University.

E. R. MULLINS, JR., Ph.D. (Illinois) Asst. Prof., Swarthmore College.

W. E. NANCE, Student, University of the South.

J. H. ORTMAN, B.A. (Wisconsin) Mathematician, Douglas Aircraft Co., Santa Monica, California.

G. K. OVERHOLTZER, Ph.D. (Indiana) Asst. Prof., Georgia Institute of Technology.

J. L. PALLONE, A.B. (Fordham) Asst. Actuary, Woodward & Fondiller, New York City.

N. C. PERRY, Ph.D. (U.S.C.) Asst. Prof., Alabama Polytechnic Institute.

W. L. PHILLIPS, JR., Student, Yale University.

T. B. PITTS, B.A. (Texas) Instr., University of Texas.

S. D. RABUSHKA, Student, Washington University.

P. A. REICHLE, M.S. (Pennsylvania) M.A. (Columbia) Col., U.S.A., retired Asst. Prof., Lenoir Rhyne College.

MRS. SUSAN L. REID, Student, Queens College.

MRS. JOY B. RUSSEK, M.S. (New York Univ.) Instr., University of Buffalo.

N. B. SMITH, M.A. (Oregon S. C.) Graduate Student, Iowa State College.

LAWRENCE SOKOLOFF, B.A. (Brooklyn) Jr. Mathematician, Republic Aviation Corp.

J. M. STARK, S.M. (M.I.T.) Graduate Student, Massachusetts Institute of Technology.

R. F. STEINHART, M.A. (Columbia) Instr., Montclair State Teachers College.

W. A. STOCK, M.A. (Northwestern) Instr., Purdue University.

C. R. STRAIN, M.S. (Purdue) Teaching Fellow, University of Michigan.

C. M. STUART, M.A. (Duke) Asst. Prof., Clemson College.

A. C. SUGAR, Ph.D. (California) Consultant, Ryan Aeronautical Co., San Diego, Calif.

F. B. TAYLOR, A.M. (Columbia) Asst. Prof., Manhattan College.

L. C. TENG, M.A. (Michigan) D.I.C. Staff, Massachusetts Institute of Technology.

E. A. TRABANT, Ph.D. (C.I.T.) Assoc. Prof., Purdue University.

C. K. TSAO, Ph.D. (Oregon) Asst. Prof., Wayne University.

J. J. URBANCEK, M.A. (Northwestern) M.S. (DePaul) Chairman, Chicago Teachers College.

W. A. VEZEAU, Ph.D. (St. Louis) Assoc. Prof., St. Louis University.

G. E. VOYLES, B.A. (North Texas S. C.) Teacher, Decatur Baptist College.

E. A. WALKER, M.A. (Sam Houston) Graduate Student, University of Kansas.

T. C. WALLACE, Student, Arizona State College, Tempe.

IRENE S. WELNA, Student, St. Joseph College.
ROY WESTWICK, Student, University of British Columbia.
E. F. WHITTLESEY, A.B. (Princeton) Instr., Bates College.
OSWALD WYLER, Sc.D. (Swiss Federal Inst. of Tech.) Asst. Prof., University of New Mexico.
MONICA J. WYZALEK, Student, Harpur College.
H. J. ZASSENHAUS, Ph.D. (Hamburg) Prof., McGill University.

THE MARCH MEETING OF THE METROPOLITAN NEW YORK SECTION

The thirteenth annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at St. John's University, Brooklyn, New York, on March 27, 1954. Professor H. F. Fehr, Collegiate Vice-Chairman of the Section, presided at the morning session and Professor W. H. Fagerstrom, Chairman of the Section, presided at the afternoon session.

One hundred two persons attended the meeting, including the following seventy-five members of the Association:

R. G. Archibald, C. Y. Bartholomew, Jonas Beraru, Samuel Borofsky, C. B. Boyer, A. B. Brown, W. F. Cassidy, G. B. Charlesworth, K. P. Clancy, Charles Clos, P. J. Cocuzza, T. F. Cope, Demetrios Counes, W. H. H. Cowles, I. A. Dodes, J. N. Eastham, W. H. Fagerstrom, H. F. Fehr, J. M. Feld, A. B. Finkelstein, A. D. Fleshler, William Forman, R. M. Foster, G. C. Francis, D. H. Frank, E. T. Frankel, E. J. Germino, William Gonzalez, Bernard Greenspan, Harriet Griffin, W. T. Hamilton, C. M. Hebbert, G. C. Helme, E. Marie Hove, Aida Kalish, G. W. Kays, L. S. Kennison, H. S. Kieval, A. E. Kinney, M. S. Klamkin, Charles Koren, Edna Kramer-Lassar, C. H. Lehmann, D. R. Lintvedt, D. M. MacEwen, May H. Maria, Emanuel Mehr, F. H. Miller, Morris Morduchow, A. J. Mortola, Eugene Odin, Maria T. Pan, J. J. Quinn, J. K. Reckzeh, R. M. Reed, H. D. Ruderman, J. P. Russell, John Salerno, Charles Salkind, A. H. Sarno, Seymour Schuster, Abraham Schwartz, Aaron Shapiro, Edward Shapiro, James Singer, Geraldine D. Smith, Morris Smith, E. R. Stabler, Mildred M. Sullivan, R. L. Swain, P. M. Treuenfels, R. M. Walter, Alan Wayne, M. E. White, R. C. Yates.

The following officers were elected for the year 1954–55: Chairman, Professor H. F. Fehr, Teachers College, Columbia University; Collegiate Vice-Chairman, Professor A. B. Brown, Queens College; High School Vice-Chairman, Mr. D. H. Frank, Forest Hills High School; Secretary, Professor E. Marie Hove, Hofstra College; Treasurer, Mr. Aaron Shapiro, Midwood High School, Brooklyn.

At the business meeting, the following report on the activities of the Committee on Contests and Awards for the year 1952–53 was given by its chairman, Professor W. H. Fagerstrom:

The fourth annual contest was held on May 14, 1953. There were 476 schools registered for the contest. Of these, 291 were in the Metropolitan New York area, and the other 185 were distributed throughout 30 states and provinces. Among the schools in this latter group were those in the units operated by British Columbia, Oregon and Western N. Y. each conducting its own state wide contest using the contest questions of the Metropolitan New York Section.

The maximum score available for the school was 450 points, and the maximum for the individual was 150 points. The three highest ranking schools were:

James Madison High School, Brooklyn, N. Y., score 349; Phillips Academy, Andover, Mass., score 336; Talmudical Academy, New York, N. Y., score 325. The three highest ranking students were: E. Alan Phillips, Weston High School, Middlesex, Mass., score 130; Jerrold Rubin, James Madison High School, Brooklyn, N. Y., score 126; Isaac J. Sharon, Talmudical Academy, New York, N. Y., score 126. There were 28 recipients of Certificates of Merit. These certificates are awarded to the highest ranking student in each area, provided his score was 85 or more. Honor keys, given in recognition of the students having won the award for two consecutive years, were awarded to 13 students.

The Committee's receipts for the year were $1687.33. Its expenses were $1680.84, leaving a balance of $6.49.

Very Reverend John A. Flynn, C. M., S. T. D., President of St. John's University, welcomed the people at the meeting, and then the following papers were presented:

1. *Mathematics in communication*, by Dr. Brockway McMillan, Bell Telephone Laboratories, Murray Hill, New Jersey. (By invitation.)

The technology of communication is divided broadly into the fields of transmission (the actual conveying of messages) and switching (the setting up of transmission paths, when needed, out of equipment common to many possible paths). Most transmission is accomplished by devices governed by linear time-invariant differential equations. The relevant applicable mathematics is then centered around the Laplace and Fourier transforms. Switching devices are discrete and quantized; their relevant mathematics is that of logic, general algebra, and combinatorics. In both fields one distinguishes between the study of apparatus *per se* and the study of its performance in the environment of use. The latter study always invokes probability theory in some form. Almost all communication problems invoke linear graphs in their statement or study.

2. *Fundamental preparation in mathematics for college study in engineering science—What the colleges need and expect*, by Professor F. H. Miller, Cooper Union.

Topics, concepts and techniques needed by the engineering student for college work in mathematics were discussed. Some of the items are essential prerequisites; others are highly desirable if the requirements of physics and engineering departments are to be adequately met. Specific examples of various topics and applications of techniques were cited, reference being made to *A Report on Mathematics Preparation for Engineering Colleges*, by F. H. Miller and S. G. Roth, published in the Journal of Engineering Education, April, 1947, and to other reports bearing on the subject.

3. *Fundamental preparation in mathematics for college study in engineering science—What the high school can do*, by Mr. D. H. Frank, Forest Hills High School.

Difficulties in meeting the needs of the engineering school are: 1) Too little mathematics taken too long before entering college, 2) too many non-essentials taught, 3) lowering of standards, 4) requirements the same for all students, 5) small numbers entering engineering resulting in difficulty in separating them from the others, 6) interest in science and engineering not necessarily accompanied by interest in mathematics.

The ways to meet these problems are: 1) separation in classes or segregation within classes, 2) integration where possible, 3) rewriting of syllabus so that each student is exposed to all the

branches of mathematics for every term of the subject, 4) discontinuation of solid geometry as separate subject and distribution over all grades, 5) weeding out all non-essentials, 6) teaching elements of differential and integral calculus in last year with stress on all necessary concepts and techniques, 7) at all times teaching for the necessary disciplines so well exemplified in mathematics.

4. *On informal symbolic logic and its place in mathematical education*, by Professor E. R. Stabler, Hofstra College.

The speaker outlined some key topics for an informal treatment of symbolic logic. These included propositions, propositional functions, quantified propositions, and classes. Then he presented examples of the potential usefulness of these topics in promoting the objectives of high school and college mathematics courses. Finally, considering the possible role of informal logic in the curriculum, he proposed: 1) an experimental twelfth grade course in logic and logical aspects of mathematics; 2) incorporation of some informal symbolic logic in college mathematics courses for general students; 3) a basic college course including informal symbolic logic, and related foundational topics, for students majoring in mathematics.

E. MARIE HOVE, *Secretary*

THE MARCH MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the Mathematical Association of America was held on March 27, 1954, at the University of Michigan, Ann Arbor, in conjunction with the meetings of the Michigan Academy of Science, Arts and Letters. Professor J. S. Frame of Michigan State University presided at both morning and afternoon sessions and at the luncheon and business meeting.

A total of one hundred thirty-two persons attended the meetings, including the following sixty-six members of the Association:

A. C. Andersen, K. J. Arnold, J. L. Bagg, J. W. Baldwin, R. C. F. Bartels, F. A. Beeler, J. H. Bell, W. S. Bicknell, J. W. Bradshaw, Fred Brafman, C. H. Butler, A. T. Butson, C. D. Calhoon, Y. W. Chen, R. V. Churchill, D. F. Coffey, S. D. Conte, J. W. Coy, C. C. Craig, D. A. Darling, P. S. Dwyer, C. M. Erikson, N. C. Fisk, K. W. Folley, J. S. Frame, J. W. Gaddum, Casper Goffman, G. W. Grotts, V. G. Grove, Frank Harary, Gerald Harrison, G. E. Hay, H. M. Heater, Fritz Herzog, T. H. Hildebrandt, E. F. Ingalls, L. G. Johnson, L. S. Johnston, P. S. Jones, Leo Katz, A. F. Lampen, J. F. Lanahan, Harry Langman, H. D. Larsen, K. B. Leisenring, J. F. Manogue, J. E. McLaughlin, L. E. Mehlenbacher, G. E. Meike, D. M. Mesner, E. E. Moise, W. K. Moore, H. W. Nace, A. L. Nelson, P. A. Nurnberger, Mary H. Payne, Emily C. Pixley, H. H. Pixley, J. H. Powell, P. H. Raker, H. F. Stelson, Brother Andrew Stephen, B. M. Stewart, J. G. Sowul, Leonard Tornheim, R. L. Wilder.

At the business meeting the nominating committee consisting of Professors H. D. Larsen, B. M. Stewart, and L. E. Mehlenbacher proposed Professor R. V. Churchill, University of Michigan, for chairman and Professor S. D. Conte, Wayne University, for Secretary-Treasurer for the year 1954–55. The slate was elected unanimously.

A motion was made by Professor R. V. Churchill and seconded by Professor V. G. Grove that the By-Laws of the Section be changed, if necessary, to allow the Sectional Governor to be a member of the Executive Committee of the Section. The motion was passed.

It was voted to empower the Executive Committee of the Section to make such resolutions as are necessary to effect the deposit and withdrawal of funds from banks.

The following papers were presented at the morning and afternoon sessions:

1. *The enumeration of connected linear graphs: an analogy to rooted trees*, by Professor Frank Harary, University of Michigan.

Using a combination of Pólya's counting theorem and a functional identity involving exponentials and the cycle indexes of the symmetric groups, it is easy to show that the method of Cayley for enumerating rooted trees is identical in form with a derivation of the number of connected linear graphs in terms of the known number of all (connected or disconnected) linear graphs.

2. *Distribution of binomial coefficients modulo p*, by Mr. D. M. Mesner, Michigan State College.

The set of $(n^2+n)/2$ binomial coefficients for exponents $0, 1, \cdots, n-1$ is considered. For p a prime, the distribution of this set into residue classes modulo p is obtained. If n has the form p^t the distribution is quite simple; in particular, the combined frequency of non-zero residues is $[(p^2+p)/2]^t$. If $f_n(i)$ denotes the frequency of numbers $\equiv i \pmod p$, then

$$\lim_{n\to\infty} f_n(i)/f_n(0) = 0; \tag{1}$$

and

$$\lim_{n\to\infty} f_n(i)/f_n(j) = 1; \qquad 0 \not\equiv i \not\equiv j \not\equiv 0,$$

It is shown that (1) holds even when p is replaced by an arbitrary modulus m.

3. *An elementary proof of an old number theorem*, by Professor Harry Langman, Detroit Institute of Technology.

The theorem states that it is impossible to have four different integral squares in arithmetic progression. The proof is entirely elementary and is based on the argument of "infinite descent." It is shown that if a set of four integers exists whose squares in order have a common difference, another set exists conforming to the same conditions and such that the largest of these is less than the largest of the assumed set.

4. *Tschebyscheff polynomial approximations in high speed computing*, Dr. Leo Razgunas, University of Michigan, Willow Run Research Center.

Presented by title.

5. *The angle sums in a simplex*, by Dr. J. W. Gaddum, Michigan State College.

In vol. 59, no. 6 of this MONTHLY, bounds were found by the speaker for the angle sums (dihedral and trihedral) in a tetrahedron. In this paper, similar bounds are discussed for the dihedral angle sum in a simplex. The bounds on a trihedral angle in terms of the surrounding dihedral angles are also discussed.

6. *Teaching of mathematics at Russian universities*, by Professor G. G. Lorentz, Wayne University, introduced by the Secretary.

Programs, teaching methods and the organization of Russian Universities were discussed.

7. *Some mathematical problems in industry*, by Dr. Morris Ostrofsky, Westinghouse Electric Corporation, Pittsburgh. (By invitation.)

Several industrial problems in mathematics were discussed, two of them in some detail. The first of these involved the numerical solution of a heat conduction equation $\partial u/\partial t = D\nabla^2 u$ + heat source where the heat source was non-linear in both time and space coordinates. It was pointed out that numerical solutions can be used as an experimental tool to crystallize physical concepts.

The second problem involved the solution of the integral equation known as the transport equation. In course of that solution numerical integrations in a form convenient for digital computers were necessary. A useful and simple formula for calculating the coefficient for numerical integration was presented. (In conclusion, the opportunities for mathematicians in industry were discussed.)

8. *Solving problems on Wayne's UDEC (Unitized digital electronic computer)*, by Dr. E. P. Little, Wayne University, introduced by the Secretary.

The Wayne University UDEC (Unitized Digital Electronic Calculator built by the Burroughs Corporation) has magnetic drum storage and five hole teletype input and output. Pulse duration is 0.1 sec. and pulse frequency is 120KC/sec. The drum stores 5300 words of 9 decimal digits plus sign and check number. Information within the machine is expressed in excess-three binary coded decimal notation. Numerical data and instructions are both stored on the drum. Random access is possible by a decision code but instructions are normally stored in sequence as far as possible. The tabulation of a simple linear function is described.

9. *On a probability distribution arising in group organization theory*, by Mr. J. H. Powell, Michigan State College.

In the very simplest model of a group organization we consider an organization of n individuals in which connections in either direction exist or do not exist between pairs. Such an organization may be represented by an incidence matrix with 0's on the principal diagonal. The immediate problem confronting us is that of the chance distribution of the number of mutual connections, *i.e.* pairs of symmetrically placed 1's in the incidence matrix. In particular, consider the distribution relative to a fixed set of row and column totals of the incidence matrix. Using a result by Katz and Powell in a paper on directed graphs, it is shown how to compute as many factorial moments of the exact distribution as desired and, implicitly, the exact probability of K mutual connections.

10. *Use of bordering in matrix inversion*, by Professor P. S. Dwyer, University of Michigan.

This paper shows how bordering, which is frequently useful in evaluating determinants, may be applied to the problem of matrix inversion. Bordering is especially useful when relations involving elements of the matrix to be inverted are present. The proof is based on the formula for the inverse of a partitioned matrix and the fact that each term of the inverse may be interpreted as the ratio of two determinants. Application is made to the problem of the size of the errors of the inverse resulting from errors in the elements and to the determination of the inverse of a correlation matrix when the factors of the reduced correlation matrix are known.

S. D. CONTE, *Secretary*

THE MARCH MEETING OF THE SOUTHEASTERN SECTION

The annual meeting of the Southeastern Section of the Mathematical Association of America was held March 19–20, 1954 at the University of South Carolina, Columbia, South Carolina. Professor W. V. Parker, Chairman of the Sec-

tion, and Professor W. L. Williams, Vice-Chairman, presided over the general sessions; Professors H. K. Fulmer, J. C. Currie, J. D. Novak, D. F. Barrow and L. A. Dye presided over subsections.

There were about two hundred in attendance including the following one hundred sixteen members of the Association:

P. L. Armstrong, Emil Artin, R. W. Bagley, D. F. Barrow, Helen Barton, R. C. Blackwell, R. G. Blake, A. T. Brauer, J. P. Brewster, G. M. Brown, J. W. Brown, B. F. Bryant, C. C. Buck, E. A. Cameron, N. A. Childress, R. S. Christian, D. H. Clanton, J. B. Coleman, J. A. Cooley, C. L. Cope, F. E. Cothran, R. W. Cowan, R. R. Croxton, J. C. Currie, N. E. Dodson, E. G. Douglas, B. M. Drucker, G. E. Duncan, L. A. Dye, E. D. Eaves, J. C. Eaves, D. O. Ellis, M. H. M. Esser, M. K. Fort, Jr., Tomlinson Fort, H. K. Fulmer, L. L. Garner, Leslie J. Gaylord, D. R. Goodner, M. O. Gonzalez, S. T. Gormsen, B. F. Hadnot, S. W. Hahn, J. C. Harden, Jr., E. A. Hedberg, Marguerite Z. Hedberg, T. F. Hickerson, T. R. Norton, L. P. Hutchison, C. W. Huff, G. B. Huff, J. B. Jackson, F. W. Kokomoor, Stephen Kulik, G. B. Lang, J. W. Lasley, Jr., C. G. Latimer, T. H. Lee, Anne L. Lewis, F. A. Lewis, G. H. Lundberg, R. A. Lytle, E. L. Mackie, C. E. Martin, S. T. Martin, W. N. Mebane, Jr., J. T. Moore, R. H. Moorman, J. C. Morelock, J. A. Nohel, J. D. Novak, G. K. Overholtzer, W. V. Parker, P. B. Patterson, A. H. Payne, Lillian G. Perkins, N. C. Perry, C. G. Phipps, R. L. Plunkett, Ellen F. Rasor, P. A. Reichle, G. E. Reves, J. M. Robertson, H. A. Robinson, F. Virginia Rohde, H. R. Rouse, W. C. Royster, W. A. Rutledge, F. W. Saunders, P. C. Scott, E. B. Shanks, Olivia H. Shanks, D. C. Sheldon, C. Eucebia Shuler, J. L. Sloan, C. B. Smith, F. W. Stallard, E. L. Stanley, L. W. Stark, C. M. Stuart, C. S. Sutton, H. S. Thurston, R. Z. Vause, Jr., J. H. Wahab, D. T. Walker, E. F. Ward, W. W. Weber, W. M. Whyburn, M. C. Wicht, W. L. Williams, N. K. Williamson, R. A. Willoughby, R. L. Wilson, F. J. Witt, J. W. Young, B. K. Youse.

The following officers were elected for the coming year: Chairman, Professor W. L. Williams, University of South Carolina; Vice-Chairman, Professor R. H. Moorman, Tennessee Polytechnic Institute; Secretary-Treasurer, Professor H. A. Robinson, Agnes Scott College.

The following program was presented:

1. *A gap in the treatment of dependent functions*, by Professor C. G. Phipps, University of Florida.

In treating dependent and independent functions a situation unaccounted for occurs where the Jacobian of the functions is only conditionally zero. This condition usually implies that one function has a maximum or a minimum subject to the restraint of fixed values for the others.

2. *Marginal notes*, by Professor J. W. Lasley, Jr., University of North Carolina.

Synthetic division of one polynomial by another, evaluations of a determinant, complex numbers as vectors, rotation of axes, the equations of conics, Newton's correction formula, derivatives of vectors, systems of linear equations, rank of a matrix are topics for which special techniques are developed.

3. *Lissajous figures in the teaching of undergraduate mathematics*, by Professor J. H. Wahab, Georgia Institute of Technology.

A method of drawing Lissajous figures is adapted to the study of parametric equations, composite functions, and the solutions of differential equations with variables separated.

4. *Mathematics in England*, by Professor Tomlinson Fort, University of Georgia.

Professor Fort spent the academic year 1952–1953 in Great Britain. He lectured at various British universities and had an unusual opportunity to see British mathematicians and to observe mathematical organizations and schools. His paper is concerned with his personal observations.

5. *A geometric device to facilitate the use of hyperbolic substitutions*, by Professor J. W. Young, University of Florida.

As a geometric device to facilitate the use of hyperbolic substitutions in integrals involving $\sqrt{u^2+a^2}$, $\sqrt{u^2-a^2}$, $\sqrt{a^2-u^2}$ and a rational function of sh ϕ and ch ϕ, Dr. Young suggested one write $\sqrt{a^2-(iu)^2}$, $i\sqrt{u^2-a^2}$, $\sqrt{a^2+(iu)^2}$, respectively and place results on a right triangle in the complex plane modified by taking the argument pure imaginary. The appropriate substitution follows by the Theorem of Pythagoras and the relations

$$\begin{matrix}\sin\\ \mathrm{sh}\end{matrix} i\phi = i \begin{matrix}\mathrm{sh}\\ \sin\end{matrix} \phi, \qquad \begin{matrix}\cos\\ \mathrm{ch}\end{matrix} i\phi = i \begin{matrix}\mathrm{ch}\\ \cos\end{matrix} \phi, \qquad \begin{matrix}\tan\\ \mathrm{th}\end{matrix} i\phi = i \begin{matrix}\mathrm{th}\\ \tan\end{matrix} \phi.$$

6. *Summer conferences in collegiate mathematics*, by Professor E. A. Cameron, University of North Carolina.

Professor Cameron told of the two 1954 summer conferences and outlined the work of the one to be held at his University.

7. *Some notes on matric computation*, by Professor W. V. Parker, Alabama Polytechnic Institute.

This paper is devoted to a discussion of various techniques in computation with matrices without resorting to the use of determinants. Several well-known results are obtained by use of these techniques.

8. *Exchange of information.*

Representatives from 28 mathematics departments participated in a discussion on certain common administrative problems which had been suggested prior to the meeting. Professor H. K. Fulmer rendered a very helpful report to the section of the findings of this discussion group.

9. *Plücker numbers and the field of values*, by Professor J. C. Currie, Georgia Institute of Technology.

Let A be a square matrix with complex elements. Then the complex numbers $xA\bar{x}'$ with the vector x subject to $x\bar{x}'=1$ fill up a convex region of the complex plane, called the field of values of A. The boundary of this region is an algebraic curve, or, more often, a part of an algebraic curve, some of whose properties are examined with the aid of the Plücker relations.

10. *Criteria for a logarithmic solution of a certain type of linear differential equation of the second order with a regular singular point*, by Professor R. W. Cowan, University of Florida.

Using the method of Frobenius, a compact expression is obtained for a general coefficient of an infinite series solution of the differential equation. The criteria desired arise from a consideration of various values of the difference of the roots of the indicial equation.

11. *On differential equations with variables separated*, by Professors J. A. Nohel and G. K. Overholtzer, Georgia Institute of Technology.

In most popular differential equations texts the problem of finding a solution of a differential equation with variables separated is treated as a mere term-by-term antidifferentiation process. In this teaching note it is pointed out that an existence theorem can be proved by elementary means.

12. *Vibrating nets*, by Professor Tomlinson Fort, University of Georgia.

Professor Fort shows how the study of a vibrating loaded net leads to the study of a boundary-value problem for a partial difference equation of the second order. The solution of these problems in their simpler forms depends upon a study of the characteristic roots of a matrix. In the more complicated cases progress has been made primarily by kinematical methods. In some cases methods similar to those used in the study of analogous problems in the theory of differential equations have proved effective.

13. *An almost regular form*, by Professor P. B. Patterson, University of Florida.

An almost regular form is defined; the form $(2, 2, 5, -1, -1, 0)$ is given as an example; and it is shown that this form does not represent primitively any m^2 where $m \equiv 1 \pmod 4$. The method of approach is essentially that used on indefinite forms by B. W. Jones and E. H. Hadlock in a joint paper presented to the Society by E. H. Hadlock on November 29, 1952.

14. *The Frenet formulas for a ruled surface*, by Professor J. D. Novak, University of South Carolina.

Extending Aoust's notion of *courbure inclinée* V. G. Grove introduced the curvature matrix and the Frenet Formulas for a ruled surface with respect to a curve. In this paper the "first and second curvatures" are used to study ruled surfaces, particularly the line of striction, and to characterize certain classes of ruled surfaces.

15. *On quasi-idempotent matrices*, by Professor G. B. Huff, University of Georgia.

Professor Huff defined a matrix A to be a quasi-idempotent if there exists a matrix polynomial $F(x)$ such that $A^r = F(r)$ for every positive integer r. It was then shown that a given matrix A is a quasi-idempotent if and only if there is an integer k such that $A(E-A)^k = 0$ and that the associated $F(x)$ is readily computed for a quasi-idempotent A. The explicit formulas lead to a miniature theory of exponentials and logarithms of certain special matrices.

16. *The numerical quartic, as a synthesis within elementary mathematics*, by Professor F. E. Cothran, Greenwood, South Carolina.

The method of Ferrari indicates parabolic and "hemi-quartic" loci which when "normalized" produce a graphical solution of the resolvent cubic. The devices used were elementary but lead the student to an interest in numerical calculus, the nature of real numbers, combinations of roots, difference and differential calculus, and hyperbolic functions, thus synthesizing several compartments of mathematics.

17. *A criterion for primary crossed products of degree p^2 over fields of formal power series in four variables*, by Professor J. T. Moore, University of Florida.

The paper provides a test for the primary nature of a crossed product algebra of the type indicated, which probably can be extended to apply to a much more general situation.

18. *A note on unstable homeomorphisms*, by Professor B. F. Bryant, Vanderbilt University.

Utz (*Unstable Homeomorphisms*, Proc. Amer. Math. Soc., vol. 1, 1950, pp. 769–774) calls a homeomorphism f of a compact, dense-in-itself, metric space X onto X unstable, with instability constant, δ, provided for each pair of points x, $y \in X$ there exists an integer $n(x, y)$ such that $d(f^n(x), f^n(y)) > \delta$. It was established that if $\omega(x)[\alpha(x)]$ does not contain periodic points then there exists u, $v \in \omega(x)[\alpha(x)]$ whose orbits are negatively [positively] asymptotic, and with the aid of this it was shown that there exist $p, q, r, x \in X$ such that the orbits of p and q are positively asymptotic and the orbits of r and s are negatively asymptotic.

19. *The theory of braids*, by Professor Emil Artin, Princeton University.

The theory of braids gives a good example that illustrates problems of topology and can also be used as an example to the concept of abstract groups. The composition of braids is done in the obvious way by tying one to the other. It can be shown that this leads to a well defined group generated by especially simple braids with several relations between them. The problem of classification amounts to a solution of the word problem in this group. The solution of the word problem can be given in two ways—an algebraic and a geometric way.

20. *An integral transform*, by Professor B. F. Hadnot, Florida State University.

The integral $J(\sigma) \equiv [\sigma^{k+1}/T(k+1)]\int_0^\infty A\lambda^k(\nu)e^{-\sigma\nu}d\nu$ for $\sigma > 0$ is proved to occur as the limit of certain (R, ϕ, κ) sums of $\sum a_n e^{-\sigma\lambda_n}$. The integral $J(\sigma)$ is viewed as an integral transform of $A\lambda^k(\omega)$ for the limit $\sigma \to 0$. One use of this idea has been to simplify certain known results in the theory of Riesz and Abel summation.

21. *Elliptic integrals in terms of Legendre polynomials*, by Professor M. O. Gonzalez, University of Alabama.

For the elliptic integral of the first kind $u = \int_0^\phi (1 - k^2 \sin^2 \phi)^{-1/2} d\phi$ the author obtains the expansion

$$u = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} P_n(\lambda) \tan^{2n+1} \frac{\phi}{2},$$

where $P_n(\lambda)$ is the Legendre polynomial of order n and $\lambda = k'^2 - k^2$.

It is also shown that

$$K = \frac{\pi}{2} P_{-1/2}(\lambda) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} P_n(\lambda)$$

and that

$$E = \frac{\pi}{4}[P_{-1/2}(\lambda) + P_{1/2}(\lambda)] = 4\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n-1)(2n+1)(2n+3)} P_n(\lambda).$$

Expansions for other elliptic integrals are also given.

22. *Necessary and sufficient conditions for convergence of infinite series*, by Professor E. B. Shanks, Vanderbilt University.

In this paper, the writer proves that most known tests for convergence or divergence of infinite series of positive terms can be deduced simply from the following two theorems.

A necessary and sufficient condition that a series $\sum a_n$ of positive terms converge (diverge) is that there exist real numbers C_n (an unbounded sequence of real numbers D_n) such that

$$C_n - C_{n+1} \geqq a_{\nu+n} \qquad (0 < D_{n+1} - D_n \leqq a_{\nu+n})$$

for some fixed $\nu \geqq 0$ and $n = 1, 2, \cdots$. For example, Kummer's test for convergence follows by taking $C_n = a_n D_n / p$, $p > 0$, $\nu = 1$ in the first theorem. Numerous other standard tests are derived similarly in the paper.

The theorems are also of theoretic importance as well as of practical importance, unlike most theorems similar in nature.

23. *Summation of series, interval of differencing one half*, by Professor F. Virginia Rohde, University of Florida.

Given a function $\phi(x)$ whose finite integral is $f(x)$, interval of differencing $h = \frac{1}{2}$. If all differences of $\phi(x)$ are zero after the kth one, an expression for $\sum_{x=1}^{n} \phi(x)$ can be found in terms of $f(x)$, $\phi(x)$, and the k differences of $\phi(x)$ taken between suitable limits.

24. *A mathematics program for high speed computation*, by Professor B. M. Drucker, Georgia Institute of Technology.

Dr. Drucker announced that his institution had received a grant for the purchase of a new electronic digital computer. He outlined the types of courses which will be offered in the fall for the preparation of programs, coding and the operation of an automatic computer.

25. *On definitions of trigonometric functions*, by Professor W. A. Rutledge, Alabama Polytechnic Institute.

The standard text-book definitions of the trigonometric functions as functions of angles are considered and some of the difficulties that arise, both technical and pedagogical, are discussed. Methods of avoiding these difficulties are presented.

26. *The proper approach to trigonometry*, by Professor Tomlinson Fort, University of Georgia.

Professor Fort has developed the subject of trigonometry with the basic idea that the infinite series is a concept with which the student is already informally familiar through his work with infinite decimals, infinite geometric series, infinite binomial series and numerous other limits in his elementary geometry. A review of these topics is proposed, still strictly in an informal manner, to be followed by a discussion of functions and the definition of $\sin x$ and $\cos x$ by infinite power series. Applications to the solution of triangles appear as one of many uses.

27. *On the least primitive root* (*mod p*), by Professor Alfred Brauer, University of North Carolina.

In a paper presented at the meeting of the American Mathematical Society at East Lansing, Michigan, on September 4, 1952, the following theorem was proved.

Let p be a prime of form $4n+1$ and k the number of different prime divisions of $p-1$. We set $2^k = r$. If g is the smallest positive primitive root (mod p), then

$$g < p^{(r-1)/r}. \tag{1}$$

In this paper the following theorem is proved which often gives a better estimate than (1).

Let $s = s(p)$ be the maximal number of consecutive integers which are all not relatively prime to $p-1$. Then

$$g < p^{s/(s+1)}. \tag{2}$$

The bounds (1) and (2) are greater than the well known bound of Vinogradoff, but are obtained by purely elementary methods.

28. *Finitely generated rings*, by Professor Emil Artin, Princeton University.

A ring R is called Noetherian if its ideals satisfy the ascending chain condition. Let R be a Noetherian ring with unit element, S an extension ring with the same unit element such that

$$S = R\omega_1 + R\omega_2 + \cdots + R\omega_n$$

for certain elements ω_ν of S. Such a ring S will be called a modul finite extension of R.

If on the other hand one can find a finite number of elements $\xi_1, \xi_2, \cdots, \xi_r$ of S such that every element of S can be expressed as polynomial of $\xi_1, \xi_2, \cdots, \xi_r$ with coefficients in R then we call S a ring finite extension of R. The following theorem is proved:

Let $R \subset T \subset S$ such that S is a ring finite extension of R but a modul finite extension of T, then T is a ring finite extension of R.

As an application one can show: A ring finite extension of a field which is a field, is a finite algebraic extension.

29. *Some inequalities concerning Mills' ratio*, by Professors D. F. Barrow and A. C. Cohen, Jr., University of Georgia, presented by Professor Barrow.

Mills' ratio is obtained by dividing the area under the normal curve to the right of any point by the ordinate of the normal curve at that point. The authors developed a method by which certain inequalities involving Mills' ratio can be more easily proved than by existing methods; and one new inequality was proved which had been only surmised before. This method depends upon the following obvious theorem:

If $F(x)$ is everywhere positive, and if the product $F(x)f(x)$ has a non-negative limit as x grows infinite and a derivative which is everywhere negative, then $f(x)$ is everywhere positive.

30. *A normalizing transformation of the fiducial distribution of point-biserial correlation*, by Professor N. C. Perry, Alabama Polytechnic Institute.

Lev has shown that r_{pbi} (point biserial correlation) has a distribution function related to non-central Student's distribution. Johnson and Welch have pointed out a normal curve approximation to the non-central 't' frequency curve. The present paper combines these two results, obtaining a useful normal approximation to a transformation of r_{pbi}.

31. *Moisture stresses in a certain type of plywood*, by Professor C. B. Smith, University of Florida.

Certain types of plywood plates having the directions of the grain of adjacent plies not perpendicular are considered. Moisture stresses produced by a uniform change in moisture content are discussed mathematically, and it is shown that by a suitable choice of the angles formed by the directions of the grain of adjacent plies warping can be avoided.

32. *Abridged series for numerical evaluation*, by Professor B. K. Youse, University of Georgia.

This paper has appeared in this MONTHLY, March, 1954, page 184.

33. *Function space approximation in laminated orthotropic material under plane strain*, by Professor R. G. Blake, University of Florida.

The method of approximation in elasticity based on the concept of function space is used to obtain a first approximation to the solution of certain problems of plane strain in a rectangle consisting of three layers of orthotropic material cemented together.

This material was prepared under a contract with the Office of Naval Research.

34. *A Cremona involutorial transformation without fundamental curves of the first species*, by Professor L. A. Dye, The Citadel.

The transformation involves a homoloidal web of cones with collinear vertices on a fundamental line of contact of the second species. The principal surfaces are images of isolated fundamental points on this line of contact.

35. *On the non-equivalence of two indefinite ternary forms*, by Professor T. R. Horton, University of Florida.

L. E. Dickson's *Studies in the Theory of Numbers* furnished a table of reduced, indefinite, non-zero, ternary quadratic forms of determinant $d \leqq 83$ except for $d=68$, $d=81$. Dickson stated (p. 147) that of four possible reduced forms of $d=68$, the equivalence of two forms A and B was not decided. This paper gives a necessary and sufficient condition that a form be equivalent to a form of a certain type and applies this condition to show A and B not equivalent.

36. *A theorem concerning mapping of a topological group into the circle*, by Professor R. L. Plunkett, Vanderbilt University.

A factor group of the group of all mappings of a compact, connected, commutative topological group, G, satisfying the second axiom of countability, into the circle is shown to be isomorphic with the character group of G. This is shown to imply that every mapping of such a group into the circle is homotopic to an interior mapping.

37. *The mapping of a second system of algebraic curves invariant under a cyclic involution of period eleven*, by Professor S. T. Gormsen, University of Florida.

In this paper the second of a series of systems of algebraic curves, invariant under an I_{11}, is mapped onto a surface F in S_7. It is found that these curves at one of the branch points on F have five tangent lines at this point. Four of these tangent lines belong to a fifth order tangent cone to F, and the fifth tangent line belongs to the remaining tangent plane to F at this point.

38. *Distinction between some geometric properties arising in non-analytic mappings*, by Professor W. C. Royster, Alabama Polytechnic Institute.

Interesting distinctions between convexity and starlikeness arise when mapping by non-analytic functions are considered. In this note the mapping function is applied to certain families of curves.

39. *On infinite series of sets*, by Professor D. O. Ellis, University of Florida.

Continuing a series of studies (Autometrized Boolean algebras, Boolean functions, Boolean metrized spaces, *etc.*) of similarities and contrasts between Boolean rings and the real field, one considers the convergence (in the Kantorovitch sequential topology) of $\sum_{i=1}^{\infty} A_i$ in a set algebra, where addition is taken as symmetric difference. The main result, quite clear after a little thought, is: $\sum_{i=1}^{\infty} A_i$ *converges if and only if* $\lim_{i\to\infty} A_i = 0$. Certain simple consequences of this result are obtained.

40. *An alternative formalism for the basic notions of algebraic topology*, by Professor C. C. Buck, University of Alabama.

The peculiarities of this notation are that the boundary of an n-cell is a set of $(n-1)$-cells, rather than a chain, and that an n-chain of a complex K over a commutative group G is written as a mapping of the n-cells of K into G. This notation exhibits clearly the relationship between the boundary chain and the coboundary of a cell.

41. *On orbital topologies*, by Professor R. W. Bagley, University of Florida.

Two problems (Hausdorff property and continuity of the topology-inducing mapping) suggested by the paper, David Ellis, *Orbital topologies*, Quart. Jour. of Math. (Oxford), vol. 4, 1953, pp. 117–119, are solved. The topological structure is determined in the special case for which the topology-inducing mapping is a permutation.

42. *Open topological disks in the plane*, by Professor M. K. Fort, Jr., University of Georgia.

A subset X of the plane has the "almost fixed point property" if corresponding to each continuous function f of X into X and each positive number ϵ there is a point p in X such that the distance from p to $f(p)$ is less than ϵ. It is shown that if an open topological disk in the plane is bounded and has a locally connected boundary, then the open disk has the almost fixed point property.

43. *Projections of surfaces obtained from an involution of period 13*, by Professor N. A. Childress, University of Florida.

The image of a planar cyclic involution of period thirteen can be represented as a surface of order thirteen in S_4. This paper concerns five successive projections of this image surface from a certain point on the surface being projected each time. Two results of a peculiar nature are explained.

H. A. ROBINSON, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The forty-first annual meeting of the Iowa Section was held jointly with the sixty-sixth annual meeting of the Iowa Academy of Science and the twenty-first convention of the Junior Academy of Science at Iowa State College, Ames, Iowa, on April 30 and May 1, 1954. The Chairman, Professor J. O. Chellevold, Professor H. T. Muhly, acting for the Vice-President, and the Secretary, Professor Fred Robertson, presided in turn.

An amendment to the by-laws was approved by a unanimous vote. The amendment reads as follows:

That Section 2 of Article II of the By-Laws of the Iowa Section of the Mathematical Association of America be replaced by:

Section 2. The Executive Committee shall consist of the officers of the Section and the Sectional Governor.

The registered attendance was sixty-eight including the following forty-four members of the Association:

E. W. Anderson, Fred Brandner, R. H. Bruck, I. H. Brune, E. L. Canfield, J. O. Chellevold, Marian Daniells, A. W. Davis, W. M. Davis, C. B. Germain, B. E. Gillam, H. E. Goheen, Cornelius Gouwens, F. S. Harper, J. J. L. Hinrichsen, D. L. Holl, R. S. Jacobsen, N. W. Johnson, Jr., G. E. Kaldenberg, A. A. Karwath, Dora Kearney, Don Kirkham, O. C. Kreider, R. J. Lambert, C. E. Langenhop, B. F. Laposky, C. H. Lindahl, W. D. Lindstrom, F. W. Lott, C. G. Maple, J. V. McKelvey, H. T. Muhly, E. N. Oberg, Fred Robertson, Hazel M. Rothlisberger, J. A. Schumaker, Augusta L. Schurrer, E. R. Smith, R. D. Stalley, H. P. Thielman, Henry Van Engen, G. P. Weeg, D. V. V. Wend, Roscoe Woods.

Officers elected for the year are: Chairman, Professor H. T. Muhly, State University of Iowa; Vice-Chairman, Professor F. A. Brandner, Iowa State College; Secretary-Treasurer, Professor Fred Robertson, Iowa State College.

The following papers were presented:

1. *Recent advances in the foundations of euclidean plane geometry*, by Professor R. H. Bruck, University of Wisconsin.

Unfortunately for the wide audience whose interest in geometry was awakened in high school or university, recent significant advances in the foundations of the subject are hidden behind a screen of abstract algebra. The present paper offers a pictorial account of some of the geometric axioms, a simple explanation of the algebraic problems which these pose and a brief account (with references rather than proofs) of the answers. The main topic is coordinatization of euclidean planes. Touched on are planar ternary rings, Veblen-Wedderburn systems, division rings with the right inverse property, right alternative and alternative division rings.

2. *Curve families and finite asymptotic values*, by Professor D. V. V. Wend, Iowa State College.

The author gives some relationships between the topological structure of a branched regular curve family F and the finite asymptotic values of the analytic functions contoured by F. Let a b-curve be an element of F separated from some other element of F by no element or chain of F. If F contains a b-curve, then any analytic function contoured by F has a finite asymptotic value. If the trees of F are isolated and F contains no b-curve, then there exists an analytic function contoured by F which has no finite asymptotic value. A precise characterization of the level curve families of harmonic polynomials is given.

3. *Peculiar derivative functions*, by Mr. N. B. Smith, Iowa State College.

Let $f(x)$ be a function defined on an interval x in E_1, and let p be a point property of $f(x)$. The function $f(x)$ is said to be peculiar with respect to the property p if there exists a partition of X into two subsets X_1 and X_2 each everywhere dense in X and such that $f(x)$ has the property p at no point of X. Peculiar derivative functions are discussed in the cases where the property p is continuity, neighborliness of W. W. Bledsoe, *Proc. Amer. Math. Soc.*, vol. 3, 1952, p. 114, and cliquishness of H. P. Thielman, this MONTHLY, vol. 60, 1953, p. 156. In particular it is shown that there exist no derivative functions peculiar on X with respect to neighborliness.

4. *Mathematical theory for the utilization of tagged atoms in determining plant nutrient transformation rates in soils*, by Professors Don Kirkham and W. V. Bartholomew, Iowa State College.

Differential equations are derived and integrated, for determining mineralization rates and immobilization (demineralization) rates, in a tagged atom system of mineral and non-mineral plant nutrients undergoing simultaneous interchange in soil. The results are

$$m = \frac{b - x}{st} L, \qquad i = \frac{x(1 - 1/s)}{t} L, \qquad L = \log \frac{b - sx_0}{b - sx},$$

$$s = \left(\frac{y_0 - y}{a} + \frac{x - x_0}{b}\right)\left(\frac{xy_0 - x_0y}{ab}\right)^{-1},$$

where m = mineralization rate; i = immobilization rate; t = time; a = mass of tagged atoms, mineral and non-mineral; b = mass of all atoms, mineral, non-mineral, tagged and untagged; x = mass (at time t) of tagged and untagged mineral atoms; y = mass of tagged mineral atoms; x_0 and y_0, values of x and y at $t = 0$. The theory is in good agreement with experiments.

5. *The gradient method for the Minkowski metric in the solution of simultaneous linear algebraic equations*, by Professors H. E. Goheen and J. R. Winkelman, Iowa State College.

The system of simultaneous equations

$$Ax - c = 0,$$

in which A is a non-singular $n \times n$ matrix and c is an $n \times 1$ matrix, can be solved by minimizing the positive definite function

$$f(x) = |\epsilon_1| + |\epsilon_2| + \cdots + |\epsilon_n|,$$

in which ϵ_i is the ith component of the $n \times 1$ matrix $Ax - c$. The authors have devised a method for obtaining this minimum.

6. *A modification of the Hartley-Politz scheme for dealing with not-at-home persons*, by Professor Alan Ross, Iowa State College, introduced by the secretary.

A method for obtaining unbiased estimates of population totals and estimates of their reliability is derived which does not require that callbacks be made on the not-at-homes in sample surveys. The theory of sampling with unequal probabilities of selection is expanded to cover the situation in which one utilizes *a priori* probabilities of obtaining schedules from all persons in the universe when these probabilities are actually determined only for those persons interviewed on first call. Horvitz and Thompson, *Journ. Am. Stat. Assoc.* vol. 47, 1952, p. 260, have considered sampling with arbitrary probabilities from the standpoint of a fixed sample size. The extension made here is to consider the case in which the sample size is fixed, but from the "sample" a variable number of schedules is obtained. Tables are presented which give comparisons of efficiency of the no-callback scheme with the usual survey technique involving repeated callbacks on the not-at-homes.

7. *The randomization analysis of a generalized randomized block design*, by Professor M. B. Wilk, Iowa State College, introduced by the secretary.

The problems of estimation of effects and of tests of significance are considered for an experimental design in which t treatments are allotted at random to r blocks, each containing pt experimental units, under the restriction that each treatment appears p times in every block. The statistical inference is referred to the actual set of experimental units employed; a linear model is used, based on means and deviations over a finite population of (conceptual) yields. No distributional assumptions are made.

The means, variances, and covariances under randomization of the analysis of variance mean squares are given. Unbiased estimates of certain linear combinations of the parameters of the model are given, together with the variances of the estimates. Attention is also given to the estimation of these variances. Tests of significance, based on a randomization test, are presented. The possible approximation to the randomization test of procedures which derive from normal theory is considered.

8. *An extension of some limit theorems in probability theory*, by Professor John Gurland, Iowa State College.

Some well-known limit theorems are proved and extended by using properties of sequences of characteristic functions. Special attention is given to stochastic convergence.

9. *Motivation in mathematics*, by Miss Ethel Cain and Students of Roosevelt High School, Des Moines, introduced by the secretary.

The speaker discussed motivation as a two-way path. The teacher may motivate the student

but the student may also motivate the teacher. Three students in the speaker's class, Mr. Jud Harper, Mr. Bob Helmick and Mr. Dick Rothrock, gave illustrations such as short hand multiplication, a method of obtaining certain combinations, a solution of a cubic equation by a system of bodies immersed in a liquid and controlled by a lever, and a set of geometric figures.

10. *A survey course for seniors*, by Professor J. O. Chellevold, Wartburg College.

In this paper the author described a senior-level course designed to show the relationship among various fields of mathematics and to provide a brief introduction to selected topics in modern mathematics.

11. *A study of student achievement and mortality in college mathematics*, by Professor Fred Robertson, Iowa State College.

The author discussed the achievements and mortality, as of the end of the winter quarter 1954, of the 1004 students who entered the college algebra course in September 1952. Success was defined as normal progress or a 2.0 or better grade point in each subject. An achievement and mortality table was shown for the group.

FRED ROBERTSON, *Secretary*

THE APRIL MEETING OF THE OHIO SECTION

The thirty-eighth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, on Saturday, April 17, 1954. Professor P. R. Rider, Chairman of the Section, presided at the morning and afternoon sessions.

Eighty-nine persons registered in attendance, including the following seventy-five members of the Association:

J. E. Adney, Jr., C. E. Amos, Grace M. Bareis, I. A. Barnett, H. M. Beatty, Foster Brooks, O. E. Brown, Emalou Brumfield, L. E. Bush, C. D. Calhoon, V. B. Caris, M. N. Chase, W. G. Clark, G. M. Clough, C. C. Crell, H. K. Crowder, R. C. Davis, B. B. Dressler, Wade Ellis, R. W. Emmert, P. L. Evans, H. E. Fettis, D. T. Finkbeiner, M. P. Fobes, H. W. Godderz, Marshall Hall, Jr., H. G. Harp, Frances Harshbarger, C. H. Heinke, R. G. Helsel, R. Y. Iwanchuk, L. A. Jehn, E. D. Jenkins, M. L. Johnson, Margaret E. Jones, John Kaiser, Chosaburo Kato, L. C. Knight, Jr., F. A. Kros, H. W. Kuhn, Nathan Lazar, R. F. Liskovec, L. L. Lowenstein, A. D. Martin, Margaret E. Mauch, S. W. McCuskey, E. J. Mickle, L. H. Miller, Knox Millsaps, Emma J. Olson, E. A. Peters, H. S. Pollard, Tibor Radó, P. V. Reichelderfer, P. R. Rider, R. F. Rinehart, D. L. Robb, G. deB. Robinson, Louis Ross, H. J. Ryser, Charles Saltzer, Samuel Selby, L. L. Shetler, R. L. Shively, Ruth B. Smyth, E. T. Stapleford, R. E. Thomas, Harold Tinnappel, H. S. Toney, W. R. Transue, D. R. Whitney, C. O. Williamson, J. A. Wilson, Alberta Wolfe, A. D. Ziebur.

The following officers were elected for the coming year: Chairman, Professor W. R. Transue, Kenyon College; Secretary-Treasurer, Professor Foster Brooks, Kent State University; third member of the Executive Committee, Professor E. J. Mickle, The Ohio State University; Program Committee: Chairman, Professor Marion Wetzel, Denison University; Professor Wade Ellis, Oberlin College; Professor H. D. Lipsich, University of Cincinnati.

The following papers were presented:

1. *Statistical distributions*, by Professor P. R. Rider, Wright-Patterson Air Force Base (Chairman's address).

Some of the statistical distributions in common use were described, methods of deriving them were discussed, and various applications were mentioned.

2. *Calculation of integrals of the form* $\int_0^\theta \sin^p \Phi \cos^q \Phi \, d\Phi$, by Mr. H. E. Fettis, Wright-Patterson Air Force Base.

Attention is directed toward the evaluation of the integral

$$\int_0^\theta \sin^p \Phi \cos^q \Phi d\Phi$$

where $p > -1$ if $\theta < \pi/2$ and $p > -1$, $q > -1$ if $\theta = \pi/2$. Exact expressions are known to be possible when $\theta = \pi/2$ for arbitrary values of p and q, and for integral values of p and q, with θ arbitrary. The present discussion shows that reduction of elliptic integrals can be effected for $p = \pm\frac{1}{2}$, $p = \pm\frac{1}{3}$, $q = 0$ and $\theta \leqq \pi/2$. Also, a rapidly converging series is given for computation with values of p and q for which closed expressions are not possible. Several practical applications are noted.

3. *On the directional derivative*, by Professor Wade Ellis, Oberlin College.

The existence of the directional derivative df/ds of the function $f = f(x, y)$ at the point (x, y) implies the existence of a plane π tangent to the surface S: $z = f(x, y)$ at the point (x, y, z). The relation between df/ds and the dihedral angle formed by π and the z-plane can be exploited to enhance the student's intuitive grasp of this and related material in courses in the calculus.

4. *n-th roots of integral binary matrices*, by Mr. Harvey Weitkamp, University of Cincinnati, introduced by Professor I. A. Barnett.

If X is a binary matrix, then X^n may be written linearly in terms of X and I by the Cayley-Hamilton equation, and the coefficients may be given explicitly in terms of the trace and determinant of X. Necessary and sufficient conditions are given in order that a binary matrix with rational or integral elements shall have an nth root with rational or integral elements. The nth roots may be unique or may be infinitely many.

5. *Linear programming*, by Dr. E. Leonard Arnoff, Operations Research Group, Case Institute of Technology. (By title.)

Linear programming, through the transportation and simplex methods, presents recently developed techniques for solving the problem of optimizing a linear functional, subject to restraining conditions which are in the form of linear inequalities. This paper first presents a brief description of the transportation and simplex methods. It then compares the types of optimization problems to which these methods are applicable with those optimization problems which can be handled by the methods usually taught in the college classroom, namely, those methods which use analytic geometry, differentiation, Lagrange multipliers, and the like.

6. *Functional mathematics for the secondary school teacher*, by Professor Emalou Brumfield, Kent State University.

The proposal is made that present mathematics-education students would be better prepared to teach secondary school mathematics if they could study, concurrently with their content mathematics courses, the importances, the applications, and the significance of these courses with respect to secondary school mathematics. The proposed study could be organized and effected by means of a mathematics-education course without affecting the content offerings of the department and without requiring the sectioning of students according to their professional objectives.

7. *The place of algebra and geometry in the undergraduate curriculum*, by Professor Gilbert deB. Robinson, University of Toronto. (By invitation.)

The place of modern algebra in the undergraduate curriculum was discussed by Professor MacLane in Baltimore last Christmas; the author here discusses integrating the algebra with the appropriate geometry. This could be done by treating *linear algebra and geometry* in a one-term course, say in the junior year; the ideas of linear transformation, lattice of subspaces of a projective space, determinants, matrices, the full linear group, *etc.*, would all be coordinated. *Quadratic ideas* involving the reduction of a quadratic form and the application of Taylor's series to a study of polarity and collineation, the inversion transformation and stereographic projection would fall naturally in the senior year. A one-term course in the first postgraduate year dealing with axiomatics, the significance of Pappus' and Desargues' theorems, the non-euclidean metric, the orthogonal group, and the groups of the regular solids in three dimensions would tie the ideas together. Such a three-term sequence could be extended and elaborated, but it would seem to provide a necessary minimum of geometrical background for a student proposing to go on to graduate work, and could replace existing courses in modern algebra and the foundations of mathematics.

8. *The convergence of sequences determined by a simple non-linear difference equation*, by Professor Samuel Goldberg, Oberlin College.

Let a real number x_0 be prescribed and determine $x_1, x_2, \cdots$ by the difference equation $x_{n+1} = x_n^2 - k$. Assume $k \geq -1/4$. The values of x_0 for which the sequence (x_n) converges are found. In the special case $k=2$, these values form a denumerably infinite set and are determined from an explicit solution of the difference equation. The mode of convergence of (x_n) to its limiting value is also discussed.

9. *Laplace transform and differential operator relations*, by Dr. L. V. Robinson, Wright-Patterson Air Force Base.

Among other relationships, it is shown that

$$\mathcal{L}[f(t)] = pf(-D_p)\frac{1}{p} = p\int_0^\infty e^{-pt}f(t)dt \qquad \left(D_p = \frac{\partial}{\partial p}\right),$$

and

$$\mathcal{L}[F(t)f(t)] = pF(-D_p)\frac{1}{p}[\mathcal{L}f(t)] = pf(-D_p)\frac{1}{p}\mathcal{L}[F(t)],$$

where $\mathcal{L}[f(t)]$ is the Laplace transform of $f(t)$.

10. *Some applications of the finite-difference analogue of Green's third identity*, by Professor Charles Saltzer, Case Institute of Technology.

The finite-difference analogue of Green's third identity is deduced from the finite-difference analogue of Green's second identity (due to Courant, Friedricks and Lewy) by the use of the fundamental solution of the Laplace difference equation developed by McCrea and Whipple, Stohr and Duffin. This identity is applied to the Laplace, Poisson, and biharmonic difference equations.

11. *How mutation geometry says "No" to the trisection problem*, by Dr. Beckham Martin, Owens-Illinois Glass Company.

Two mutation views were developed: one starting from two prototypes resulting in the mutation trisection expression

$$2b \cdot rd \cdot r = (pb - sd) \cdot r$$

where b and d are the orthonormalized bisectors of the given angle and its supplement. The equation was transmuted to a primordial prototype

$$A \cdot r = 0,$$

whose solution was

$$r = 0/0.$$

A second view started from a single prototype which led to a factorable 4th degree equation with a free choice parameter with which to control its irreducibility. A mutation curve was drawn for this which was shown to trisect the angle.

FOSTER BROOKS, *Secretary*

THE APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The thirty-seventh annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at Colorado Agricultural and Mechanical College, Fort Collins, Colorado, on Friday afternoon and evening and Saturday forenoon, April 30 and May 1, 1954. Professor M. L. Madison, Chairman of the Section, presided at all three sessions.

Seventy-five registered for the meeting, including the following forty-nine members of the Association:

C. F. Barr, D. L. Barrick, J. R. Britton, R. G. Buschman, R. K. Butz, F. M. Carpenter, A. G. Clark, Sarvadaman Chowla, G. S. Cook, Rev. F. T. Daly, W. E. Dorgan, H. T. Guard, R. R. Gutzman, C. L. Harbison, Leota C. Hayward, I. L. Hebel, LeRoy Holubar, J. E. Householder, P. F. Hultquist, C. A. Hutchinson, B. W. Jones, A. J. Kempner, Claribel Kendall, R. B. Kriegh, J. S. Leech, M. L. Madison, W. E. Mientka, M. W. Milligan, W. K. Nelson, Greta Neubauer, D. O. Patterson, O. M. Rasmussen, O. H. Rechard, A. W. Recht, L. W. Rutland, Jr., Nathan Schwid, S. R. Smith, W. N. Smith, L. C. Snively, K. H. Stahl, J. McD. Staley, P. O. Steen, E. P. Tovani, E. L. Vanderburgh, W. W. Varner, J. F. Wagner, F. J. Wall, C. R. Wylie, Jr., A. Zirakzadeh.

Officers elected at the meeting for 1954–1955 were: Chairman, Professor Nathan Schwid, University of Wyoming; Vice-Chairman, Professor C. R. Wylie, Jr., University of Utah; Secretary-Treasurer, Professor F. M. Carpenter, Colorado School of Mines.

The following papers were presented:

1. *On expressing the matrix A^t as a polynomial in t,* by Professor R. K. Butz, Colorado Agricultural and Mechanical College.

This paper discusses the notion of $E_\alpha F(X)$ matrices as introduced by G. B. Huff (*Matrices such that A^t is a polynomial in t and principal idempotent elements*, Bull. Amer. Math. Soc., vol. 59, 1953, p. 54). Emphasis is placed on the fact that the proofs of the main theorems require only the more elementary concepts of matrix theory and on methods of finding $F(X)$ given a matrix A with elements in the field of complex numbers. The clarity with which some classical results follow by the use of this notion is pointed out.

2. *An optimum solution of N equations in M unknowns with N greater than M,* by Mr. Leon Rutland, University of Colorado.

A problem in engineering design led to a consideration of the system of equations

$$\sum_{j=1}^{m} a_{ij}x_j = l_i \qquad (i = 1, 2, \cdots, n), (n > m),$$

where the desired optimum solution of the system is that set of x's for which the largest absolute value of any of the deltas in the set of equations

$$\Delta_i + \sum_{j=1}^{m} a_{ij}x_j = l_i, \qquad (i = 1, 2, \cdots, n), (n > m),$$

is as small as possible. Several theorems giving solutions to the problem under various conditions were either proved or stated with the proofs being omitted. A numerical example, carried through on a digital computer, was cited to indicate that even though there are as many as forty equations the method is feasible and the answer can be readily attained.

3. *The method of Frobenius*, by Professor R. H. Cook, South Dakota School of Mines and Technology, introduced by the secretary.

The usual textbook presentation of the method of Frobenius effectively camouflages two important points: (1) that the method involves a Taylor's expansion, and quite often leads to just a Taylor's expansion; (2) the conditions under which the method is applicable. This paper suggests a modified approach which has neither of the above disadvantages, is easily taught, and is sufficiently flexible to be applicable to many non-linear problems.

4. *Approximate solutions to a certain functional equation*, by Professor C. A. Rogers, Colorado A and M. College.

The following is investigated: Given a non-negative $g(x)$, defined for all $x>0$, and which is strictly monotonic increasing and everywhere differentiable, to find a closed-form $f(x)$, reasonably computable, such that $f[f(x)]$ is at least approximately equal to $g(x)$. It was indicated how this approximation could be accomplished for certain g-functions, with examples.

5. *Some infinite series*, by Dr. W. E. Briggs, Professor S. Chowla, Professor (Emeritus) A. J. Kempner, and Research Assistant W. E. Mientka, University of Colorado, presented by Mr. Mientka.

It is proved that

$$\sum_{1}^{\infty} \frac{\sigma_n}{n^3} = 2\zeta(3),$$

where

$$\sigma_n = \sum_{l=1}^{n} \frac{1}{l}.$$

6. *Effect of rotation on the normal mode frequencies of transverse vibration of a cantilever beam*, by Professor R. H. Cook and Mr. L. J. Eatherton, South Dakota School of Mines and Technology, presented by Mr. Eatherton.

The differential equation,

$$EI\frac{\partial^4 y}{\partial x^4} - \frac{\partial^2 y}{\partial x^2}\int_x^L \rho A S\Omega^2 dS + \rho A x\Omega^2 \frac{\partial y}{\partial x} + \rho A \frac{\partial^2 y}{\partial t^2} = 0,$$

describes the vibrations, in a vertical plane, of a cantilever beam which rotates about a vertical axis through its clamped end. This equation and appropriate boundary conditions are considered

by the use of Taylor's expansion. The normal mode frequencies are calculated in terms of Ω, the rotational speed. Results show the dependence upon both Ω^2 and Ω^4 for the first and second modes and upon Ω^2 for the third mode. They are in excellent agreement with existing experimental data.

7. *Block designs*, by Professor Burton W. Jones, University of Colorado.

The definition and significance of balanced incomplete block designs are briefly described and methods of exclusion sketched.

8. *The mapping of the circles of S_2 into the points of S_3*, by Professor C. R. Wylie, Jr., University of Utah. (By invitation).

If the coefficients a, b, c, d in the general equation of a circle

$$d(x^2 + y^2) - 2ax - 2by + c = 0$$

are interpreted as homogeneous point coordinates in S_3, point circles are mapped into points on the paraboloid

$$V \equiv a^2 + b^2 - cd = 0,$$

proper real circles are mapped into finite points outside V, improper real circles (lines) are mapped into real points at infinity, and imaginary circles are mapped into finite points within V. Pencils and bundles of circles are represented in S_3 by lines and planes, respectively, and may be classified according to the intersection of their images with V. Two circles which are orthogonal are represented by points each of which lies in the polar of the other with respect to V. Conjugate pencils of circles are represented by lines conjugate with respect to V. Various theorems from college geometry were interpreted in S_3, and the classical constructions for circles satisfying three conditions were considered as problems in descriptive geometry in S_3.

9. *The University of Colorado Engineering Experiment Station analog computer—The UCEESAC*, by Mr. Walter W. Varner, University of Colorado.

A description of the Boeing analog computer recently installed at the University of Colorado was given. Types of problems that can be solved as well as restrictions on their solution were given. Finally a simple pair of simultaneous differential equations were considered and the simplicity of forming the wiring diagram shown.

10. *Fitting empirical equations to fluid meter data*, by Professor S. R. Smith, University of Wyoming.

Empirical equations of the form

$$C = \frac{R}{a + bR} + de^{f/R}*$$

were fitted to both flow nozzle and orifice meter data and residuals determined. R, C curves were fitted to data for the fluids stream, oil and water for $0 < R \leqq 3{,}040{,}000$. C is the coefficient of discharge of the meter and R its corresponding Reynolds number, both dimensionless.

11. *On $ff(x) = F(x)$, $F(x)$ given (real), $f(x)$ unknown*, by Professor (Emeritus) A. J. Kempner, Dr. W. E. Briggs, Professor S. Chowla and Mr. W. E. Mientka, University of Colorado, presented by Professor Kempner.

For the real case the geometrical interpretation employed in studying $f(x) - x = 0$ can be extended so as to lead immediately to results such as: there exist totally discontinuous functions $f(x)$

* Equation first used by I. D. Ruggles, former graduate student in mathematics, University of Wyoming.

for which $ff(x)$ is single-valued and continuous; or, the function $y=g(x)$ given by $x-y+\pi/2 = \mu \cos(x+y)$, $|\mu| \leq 1/2$, is a single-valued inverse iterate of $f(x)=x+\pi$. It also makes plain the plausibility of introducing iterations of $f^{(n)}(x)$ of any rational (or even any real) index n.

12. *Figure it out for yourself*, by Professor A. W. Recht, University of Denver.

Trained mathematicians are at a premium; everyone realizes the great part mathematics plays in a technical civilization. Yet election of mathematics in high schools is waning, despite efforts of government and mathematical groups stressing urgent need of training of the talented. The problem is to reach the 85% who will use mathematics in normal pursuits and also train the talented 15%. The paper suggested revisions of policy, including training of really professional mathematics teachers, emphasis on students doing own work, insistence on 100% accuracy, promotion of "first the problem, then the mathematics," and strengthening of fundamental concepts to avoid mathematical accidents in home and industry.

13. *Textbooks on elementary mathematics*, by Professor J. S. Leech, Colorado College.

Attention is called to the fact that in a large majority of textbooks many terms are defined erroneously or ambiguously. Many theorems are stated and "proved" without any or with incomplete hypotheses. In all of these cases, the author believed that the correct definitions and statements of theorems not only do not increase the difficulty of the subjects, but serve to clarify concepts and contribute greatly to understanding.

14. *Freshman mathematics separation*, by Professor I. L. Hebel, Colorado School of Mines.

This paper dealt with the mathematics department's experience over the last seven years in assigning and classifying freshman mathematics students.

F. M. CARPENTER, *Secretary*

THE APRIL MEETING OF THE SOUTHWESTERN SECTION

The fourteenth annual meeting of the Southwestern Section of the Mathematical Association of America was held at Arizona State College, Tempe, Arizona, on April 16, 1954. Professor M. S. Hendrickson, Chairman of the Section, presided.

Forty persons attended the meeting including the following twenty-seven members of the Association:

O. B. Ader, C. E. Aull, C. E. Buell, J. H. Butchart, D. G. Duncan, J. F. Foster, Jr., R. S. Fouch, F. C. Gentry, R. F. Graesser, R. E. Graves, W. P. Heinzman, M. S. Hendrickson, Carol Karp, Max Kramer, Lincoln LaPaz, R. B. Lyon, W. W. Mitchell, Jr., E. D. Nering, J. L. Olpin, E. J. Purcell, L. C. Snively, A. H. Steinbrenner, Deonisie Trifan, Earl Walden, D. L. Webb, Charles Wexler, Oswald Wyler.

The following were elected officers for the year 1954: Chairman, Professor D. L. Webb, University of Arizona; Vice-Chairman, Professor R. L. Westhafer, New Mexico College of Agriculture and Mechanic Arts; Secretary-Treasurer, Professor W. W. Mitchell, Jr., Phoenix College (four year term); Lecturers, Professor Max Kramer, New Mexico College of Agriculture and Mechanic Arts (one year), Professor D. G. Duncan, University of Arizona, (two years).

An invited address "The Story of Palomar" was given following the banquet by Dr. S. B. Nicholson of the Mount Wilson and Palomar Observatories. Dr. Nicholson also read a paper on "Results from Palomar" during the morning session.

The following papers were presented:

1. *Some problems in electrostatics treated synthetically*, by Professor J. H. Butchart, Arizona State College.

The author showed that synthetic methods, notably inversion, are appropriate in problems concerning electrostatic forces in the plane. He discussed in particular the case of concyclic charges.

2. *Nonlinear differential equations that can be reduced to linear*, by Professor M. S. Hendrickson, University of New Mexico. (By title.)

The most general nonlinear equation of the first order which is reducible by a substitution of the form $u=\phi(x, y)$ to the linear equation $u'=up(x)+q(x)$ is of the form $y'=-(\phi_x/\phi_y)+(1/\phi_y)[p\phi+q]$. In particular, any equation of the form $y'=h(y)[F(x)+G(x)\exp -\int 1/hdy]$ is reducible by the substitution $u=\exp[\int 1/hdy-\int Fdx]$ to the equation $u'=G(x)\exp -\int Fdx$ which can be solved by integration alone.

The most general second order equation reducible by a substitution of the form $F(y')=G(y)$ to a first order equation linear in G with y as independent variable is of the form $y''+p(y)F(y')y'/F'(y')+q(y)y'/F'(y')=0$. In particular, any equation of the form $y''+g(y)(y')^2+h(y)(y')^{2-n}=0$ can be reduced by the substitution $G(y)=(y')^n$.

3. *Set representation theorems in implicative models*, by Mrs. Carol R. Karp, New Mexico College of Agriculture and Mechanic Arts.

Implicative Models were introduced by L. Henkin in his paper *An algebraic characterization of quantifiers*, Fundamenta Mathematicae, 1950. They are the algebraic models for the System of Basic Implication, a calculus of propositions with only implication as a propositional connective, yielding as formal theorems only *wffs* valid in both the classical and intuitionistic calculi of propositions.

The result announced was that every Implicative Model $\Psi=<X, 0, \dot{-}>$ can be extended to a Brouwerian Algebra with preservation of infs and exactly those tops which satisfy the following conditions: (i) $z=$top Y in the sense of Henkin, *i.e.*, (1) z is an upper bound of Y; (2) for any z', a in X, $y\dot{-}a\leqq z'$ for all y in Y implies $z\dot{-}a\leqq z'$. (ii) for any z', $a_1, a_2, \cdots, a_n$ in X, n any finite integer, $y\dot{-}a_1\dot{-}a_2\dot{-}\cdots\dot{-}a_n\leqq z'$ for all y in Y implies $z\dot{-}a_1\dot{-}\cdots\dot{-}a_n\leqq z'$.

By a result of Tarski and McKinsey [*Closed elements in closure algebras*, Annals of Mathematics, 1946] it follows that every Implicative Model is isomorphic to a subspace of closed sets of a topological space.

4. *Axioms of congruence for absolute geometry in a bounded domain*, by Professor Oswald Wyler, University of New Mexico.

A system of axioms of congruence is given which is valid in any open convex domain (whether bounded or not) of a euclidean, hyperbolic or elliptic geometry. Together with axioms of incidence and of order given elsewhere by the author (*Duke Math. Jour.* vol. 20, 1953; *Compositio Math.*, vol. 11, 1953), and with an axiom of continuity, these axioms permit the construction of the full absolute geometry from the geometry of any open convex domain. No axiom of parallelism is used, but the type of a geometry (euclidean, hyperbolic, or elliptic) is given by properties of its absolute polarity.

5. *A method of approximating moments in thin elastic plates*, by Professor Deonisie Trifan, University of Arizona.

A method of approximating the bending and twisting moments in thin elastic was presented. Based on a theory developed by Prager and Synge who regarded a state of stress in a deformed body as a point in function space, the actual solution which satisfies the equation of equilibrium, compatibility conditions, and boundary conditions is approximated by a series of so-called artificial states which satisfy only some of the above conditions.

6. *The case for intuitionism*, by Professor E. J. Purcell, University of Arizona.

7. *A three dimensional model of a lattice*, by Professor D. L. Webb, University of Arizona.

The display of a symmetric model in three dimensions of a free modular lattice of three generators.

8. *On normal subgroups of even order*, by Professor K. A. Fowler, University of Arizona, introduced by the secretary.

Let G be a finite group of even order n which contains exactly m elements of order 2. The following results are obtained: (1) $h-1 \geq (m^2+m)/n$ where h is the number of conjugate classes each of which is equal to its inverse class; (2) there exists a real character, distinct from the 1-character, of degree less than m/n; (3) G properly contains a normal subgroup of bounded index, the bound depending only on the ratio m/n.

9. *Families of lines*, by Dr. Andrew Sobczyk, Los Alamos Scientific Laboratory.

The family of all lines in the plane is in one-to-one correspondence with the Möbius cylinder, since, as may be seen from the normal form, each line is characterized by two coordinates p, α with $-\infty < p < \infty$, $0 \leq \alpha < \pi$ and the boundary $\alpha = \pi$ of the p, α strip must be identified into the boundary $\alpha = 0$ with reversal of the sign of p. The family of all lines in space is in one-to-one correspondence with the fiber-bundle of the tangent planes to a sphere. Many new and interesting results on line families are obtained by study of their graphs in these fiber bundles.

10. *Coalition formation in n-person games*, by Professor E. D. Nering, University of Minnesota and Goodyear Aircraft Company.

The von Neumann and Morganstern theory of n-person games is essentially an extrapolation of an intuitive analysis of the zero-sum 3-person game. Except for this case the theory gives little information about the process of formation of coalitions nor, on the other hand, is the effect of such a process on the outcome of the game analyzed. The author presents an intuitive analysis of the zero-sum 4-person game which is an extension of the analysis of the zero-sum 3-person game and which leads to results different from those given by the von Neumann and Morganstern theory.

11. *Mathematics grades of engineers*, by Professor F. C. Gentry, University of New Mexico.

A survey of mathematics grades of a selected group of engineering graduates compared to their grades in other subjects was discussed. It was found that on the whole grades in mathematics are slightly lower than overall grade averages, that very few of these students failed a first year course, but about 25% failed one course in calculus.

R. L. WESTHAFER, *Secretary*

EMPLOYMENT OPPORTUNITIES

Mathematics Research, Inc., State College, Pa. Faculty consultant on industrial machinability research project, mathematical analysis of drilling, *etc.*

The MONTHLY is devoting this space to paid announcements of employment opportunities for mathematicians. The text of such announcements should be in want-ad form and must be in the hands of the editor (C. B. Allendoerfer, Mathematics Department, University of Washington, Seattle 5, Wash.) before the first day of the month preceding the issue in which the notice is to appear. Announcements should indicate the academic rank or similar description of the opening, but should not mention a specific salary. Blind ads are permissible which direct replies to a specific box number in care of the Mathematical Association of America, Buffalo 14, N. Y. In order to conserve space and achieve uniformity, the privilege is reserved to rearrange advertisements. Advertisers will be billed at the rate of $1.50 per line. Rates for display advertising may be obtained from the Advertising Manager.

CALENDAR OF FUTURE MEETINGS

Thirty-eighth Annual Meeting, University of Pittsburgh, Pittsburgh, Pennsylvania, December 30, 1954.

Thirty-sixth Summer Meeting, University of Michigan, Ann Arbor, Michigan, August 29–30, 1955.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pennsylvania, May, 1955.

ILLINOIS, Monmouth College, Monmouth, May 13–14, 1955.

INDIANA, Butler University, Indianapolis, May, 1955.

IOWA, St. Ambrose College, Davenport, April 15–16, 1955.

KANSAS

KENTUCKY

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Georgetown University, Washington, D. C., December 4, 1954.

METROPOLITAN NEW YORK, Queens College, Flushing, New York, April 30, 1955.

MICHIGAN, Michigan State College, East Lansing, Spring, 1955.

MINNESOTA, University of Manitoba, Winnipeg, October 16, 1954.

MISSOURI, University of Kansas City, Spring, 1955.

NEBRASKA

NORTHERN CALIFORNIA, University of California, Berkeley, January 15, 1955.

OHIO

OKLAHOMA, Oklahoma City University, October 29, 1954.

PACIFIC NORTHWEST

PHILADELPHIA, Princeton University, Princeton, New Jersey, November 27, 1954.

ROCKY MOUNTAIN, University of Wyoming, Laramie, Spring, 1955.

SOUTHEASTERN, Tennessee Polytechnic Institute, Cookeville, March 11–12, 1955.

SOUTHERN CALIFORNIA, Santa Monica City College, March 12, 1955.

SOUTHWESTERN, University of New Mexico, Albuquerque, Spring, 1955.

TEXAS, Abilene Christian College, Abilene, April, 1955.

UPPER NEW YORK STATE, University of Buffalo, May 14, 1955.

WISCONSIN, Cardinal Stritch College, Milwaukee, May, 1955.

New opportunities for

MATHEMATICIANS

as
Applied Science
Representatives
with

IBM

These are excellent positions requiring top level work as consultants and educators in the application of IBM electronic computing equipment to the most challenging problems of business and science.

The opportunities for growth in this vast new field are exciting.

Employment assignments are available in major cities throughout the United States.

Good salaries, excellent working conditions, exceptional employee benefits.

Minimum requirements: Major or graduate degree in Mathematics, Physics, or Engineering with Applied Mathematics equivalent.

Desirable, but not required: Previous experience in teaching applied mathematics and use of automatic computing equipment.

Write, giving full details, including experience and education, to

Dr. C. C. Hurd, Director
Applied Science Division
Room 102
INTERNATIONAL BUSINESS MACHINES
590 Madison Avenue
New York 22, N. Y.

Up-to-date mathematics texts from Ginn

ADVANCED CALCULUS (in press)

by Angus E. Taylor,
University of California at Los Angeles

ANALYTIC GEOMETRY AND CALCULUS

by William R. Longley, Percey F. Smith,
and Wallace A. Wilson

COLLEGE ALGEBRA AND PLANE TRIGONOMETRY

by James H. Zant,
Oklahoma A. and M. College

Home Office:
Boston

Sales Offices: New York 11 Chicago 16 Atlanta 3
Dallas 1 Columbus 16 San Francisco 34 Toronto 7

MATHEMATICS AND PLAUSIBLE REASONING

By GEORGE PÓLYA

A two-volume guide to the practical art of plausible reasoning by the famous author of *How to Solve It.* Professor Pólya uses mathematics to show how hunches and guesses play an important part in even the most rigorously deductive sciences.

Volume I. Inductions and Analogy in Mathematics 336 pages. $5.50
Volume II. Patterns of Plausible Inference 240 pages. $4.50
The set $9.

Order from your bookstore, or

PRINCETON UNIVERSITY PRESS, Princeton, New Jersey

Special Sale

OF VOLUMES OF

THE AMERICAN MATHEMATICAL MONTHLY

•

Complete volumes from volume 23 (1916) to volume 52 (1945) inclusive, are now available at the following rates:

Single volumes:	$10 per volume
Five or more volumes (any years):	$ 5 per volume

We pay transportation charges if payment accompanies order.

•

Harry M. Gehman, Secretary-Treasurer
Mathematical Association of America
University of Buffalo
Buffalo 14, New York

Macmillan Texts

DIFFERENTIAL AND INTEGRAL CALCULUS

fifth edition

by Clyde E. Love and Earl D. Rainville

—An important revision of many of the old topics with the following new topics added:

(1) Work (2) Circle of curvature (3) Integral test for infinite series (4) Summation of power series (5) Oblique and curvilinear asymptotes (6) Evaluation of iterated integrals by inversion of order (7) Evaluation of iterated integrals by change of coordinate system (8) Short appendix on rigorous presentation of limits.

1954 526 pages $5.75

TRIGONOMETRY by John F. Randolph

Using the "general angle" approach to the subject, Dr. Randolph has written a trigonometry text adaptable to courses with emphasis ranging from the simplest numerical work to modern stress on analytical trigonometry. He presents the subject in such a way that the student not only learns facts about trigonometry but also obtains experience in sound thought processes.

1953 220 pages $3.00

The Macmillan Company

60 FIFTH AVENUE, NEW YORK 11

PLANE TRIGONOMETRY

By GORDON FULLER, Alabama Polytechnic Institute. 270 pages, (with tables), $3.25; (text alone), $2.75

Here is a work designed as a standard text for courses in all college mathematics departments. The author covers both analytical and numerical trigonometry and avoids explanations that are too brief. Each new topic presented is illustrated with problems worked out in detail.

GENERAL COLLEGE MATHEMATICS

By W. L. AYRES, C. G. FRY, and H. F. S. JONAH, Purdue University. 288 pages, $3.75

A new approach to freshman mathematics, designed to help the student in his application of mathematical topics to the biological and social sciences. Keyed primarily for those students who will take only one year of college mathematics, this text examines such subjects as ratios and percentages, linear and quadratic equations, trigonometry, interest and its application to installment buying, laws of growth, statistics, etc.

CALCULUS AND ANALYTIC GEOMETRY

By C. T. HOLMES, Bowdoin College. 416 pages, $5.00

Designed for a combination course in which the concepts and techniques of the calculus are the main objectives. Although calculus is emphasized, the essentials of analytic geometry are presented in sufficient detail for a subsequent major. The concept of integration is introduced early in the text.

DIFFERENTIAL EQUATIONS

By ROBERT C. YATES, United States Military Academy. 215 pages, $3.75

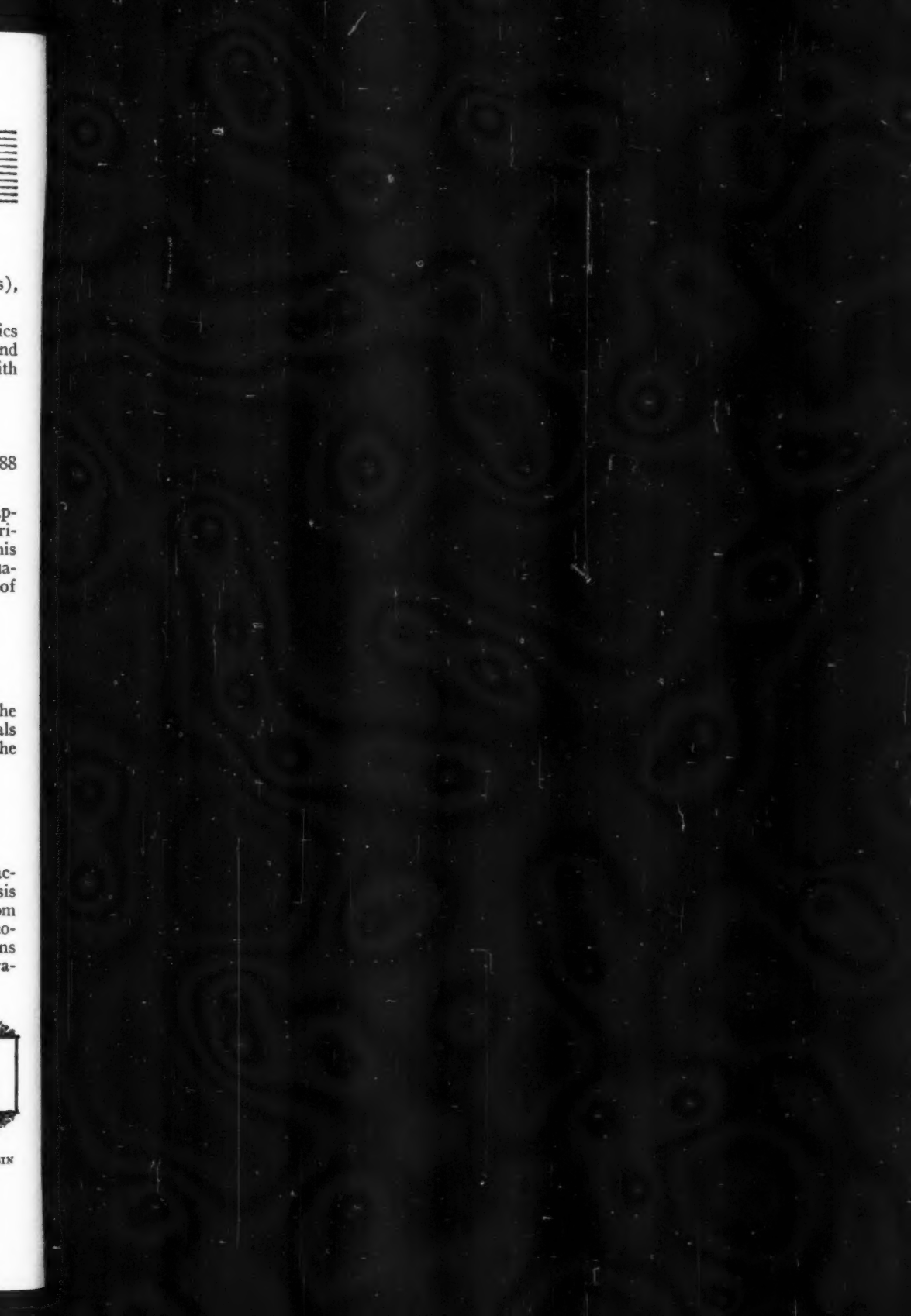
This text is designed to prepare the student for work in modern engineering practice and theory and to present the basic mathematical tools necessary for analysis and solution of problems leading to differential equations. The book ranges from first order equations through general linear ordinary equations with constant coefficients, special equations mostly of a nonlinear character, numerical solutions and solutions by series, and then into the solution of the wave equation by separation of variables.

Send for copies on approval

McGRAW-HILL BOOK COMPANY

330 West 42nd Street • New York 36, N. Y.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN



TECHNOLOGY DEPARTMENT

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 61 NUMBER 7

PART II

PUBLIC LIBRARY
SEP 20 1954

Proceedings of the

Symposium on Special Topics in Applied Mathematics

HELD AT
NORTHWESTERN UNIVERSITY, NOVEMBER 27–28, 1953

Number 3
of the
HERBERT ELLSWORTH SLAUGHT
MEMORIAL PAPERS

AUGUST-SEPTEMBER 1954

ONE DOLLAR A COPY

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

Carl B. Allendoerfer, *Editor*

ASSOCIATE EDITORS

R. V. CHURCHILL	EDWIN HEWITT	GEORGE SPRINGER
HOWARD EVES	L. M. KELLY	E. P. STARKE
F. A. FICKEN	IVAN NIVEN	G. B. THOMAS
WALLACE GIVENS	EDITH R. SCHNECKENBURGER	E. P. VANCE

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. Allendoerfer, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. Randolph, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

OFFICERS OF THE ASSOCIATION

President, E. J. McShane, University of Virginia
Honorary President, W. D. Cairns, Oberlin College
First Vice-President, H. S. M. Coxeter, University of Toronto
Second Vice-President, W. L. Duren, Jr., Tulane University
Secretary-Treasurer, H. M. Gehman, University of Buffalo
Associate Secretary, Edith R. Schneckenburger, University of Buffalo
Editor, C. B. Allendoerfer, University of Washington

Additional Members of the Board of Governors: I. A. Barnett, M. A. Basoco, B. H. Brown, S. S. Cairns, H. H. Downing, L. R. Ford, R. M. Foster, Philip Franklin, R. F. Graesser, E. R. Heineman, R. C. Huffer, Ralph Hull, C. A. Hutchinson, S. B. Jackson, R. D. James, L. W. Johnson, P. S. Jones, R. E. Langer, D. H. Lehmer, F. A. Lewis, Saunders MacLane, K. O. May, R. J. Michel, E. B. Miller, W. E. Milne, C. W. Munshower, W. H. Myers, C. O. Oakley, Morris Ostrofsky, G. B. Price, F. A. Rickey, C. E. Springer, C. W. Trigg, A. W. Tucker

---

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

Proceedings of the

SYMPOSIUM ON SPECIAL TOPICS IN APPLIED MATHEMATICS

HELD AT

NORTHWESTERN UNIVERSITY, NOVEMBER 27–28, 1953

Sponsored by the Mathematics Division of the
National Research Council under contract
with the National Science Foundation
and the American Mathematical
Society

The Third
HERBERT ELLSWORTH SLAUGHT
MEMORIAL PAPFR

This publication has been financed jointly by the National Science Foundation and the Arnold Buffum Chace Fund of the Mathematical Association of America

Published as a supplement to the AMERICAN MATHEMATICAL MONTHLY

Volume 61 Number 7

August–September, 1954

TABLE OF CONTENTS

Introduction and Critique . . . . . F. J. WEYL 1

The Physical Facts and the Differential Equations . . . JEAN LERAY 5

Recent Advances at Stanford in the Application of Conformal Mapping to Hydrodynamics. . . . . P. R. GARABEDIAN, EDWARD MCLEOD, JR., AND MARTIN VITOUSEK 8

Singularities on Shocks . . . . . A. H. TAUB 11

Signal and Noise Problems . . . . . M. KAC 23

Boolean Algebras in Electric Circuit Design . . . . D. E. MULLER 27

Computational Attacks on Discrete Problems . . . S. S. CAIRNS 29

On Characteristic Value Problems in High Order Differential Equations Which Arise in Studies on Hydrodynamic and Hydromagnetic Stability . . . . . S. CHANDRASEKHAR 32

Frequency Spectrum of Vibrations of a Crystal Lattice . . . . . ELLIOTT W. MONTROLL 46

INTRODUCTION AND CRITIQUE

F. J. WEYL, Investigator, NRC Committee on Training and Research in Applied Mathematics

1. Sponsorship and Organization. The papers appearing here under a single cover were presented in the course of a somewhat novel type of symposium on special topics in applied mathematics which was organized by the Mathematics Division of the National Research Council and accommodated by the American Mathematical Society as an integral part of its regular meeting at Northwestern University, Evanston, Illinois, on November 27–28, 1953. The character of this symposium was largely set by the fact that its organization constituted one of the phases of a Survey of Training and Research in Applied Mathematics which the National Research Council has been conducting during the past year under provisions made by the National Science Foundation jointly with the three military Offices of Research in response to a recognized need for a comprehensive stock-taking of this country's resources in the way of mathematics and mathematicians likely to play a significant part in the future advances of our science and technology.

The arrangement of one or several conferences on the various phases of training and research in applied mathematics formed part of the plans for the Survey from the beginning. As subsequent work brought the picture better into focus, it became clear that the modern range and depth of the subject matter of applied mathematics was as yet insufficiently understood by many mathematicians. Today applied mathematics is emphatically no longer just the art of solving by dull methods repulsive differential equations whose coefficients carry physical or engineering names! Rather than merely pointing out the necessity of a systematic effort to dispel this deadly myth, the NRC Committee on Training and Research in Applied Mathematics which was in charge of the Survey decided to take at once an active step in this direction. Hence one of the planned conferences was set aside for this particular purpose and developed into the Symposium on Special Topics in Applied Mathematics here reported.

2. Aim and Impact. The Symposium was designed to present characteristic current research to a large audience of mathematicians, illustrating particularly active sectors of the front along which mathematics interacts today with other scientific disciplines. Its philosophy differed therefore markedly from the Applied Mathematics Symposia convened annually by the Mathematical Society or the younger series held under Office of Ordnance Research sponsorship. No attempt was made, as in the latter, to aim the program at the workers and specialists in some unified area, however broadly defined, and the interchange of information among them. Instead, the emphasis was on diversity of topics, on highlighting some exciting developments of the day and doing it in a manner to induce broad understanding among the mathematically literate. What we wanted was a collection of significant examples, not brought as yet too often or

accessibly to the attention of the mathematicians, which would show just how varied and substantial the contributions of the genuinely secular mathematician (as distinct from the monastic one) can be in this interaction of his modes of thought and analysis with scientific endeavors in other fields.

The Symposium was therefore planned in three 90-minute sessions of three half-hour papers each. The first of these sessions was to be concerned with new mathematical theories and techniques which are currently being fashioned for applied purposes; the second one was to be dedicated to the characteristic mixture of combinatorial analysis and probability theory now being developed for the sake of constructing analytic models and the numerical simulation of control and communication systems; the third one, finally, was to deal with situations where the interplay between physical ideas and mathematical construction is at present the critical element. This structure of the sessions is still discernible in the Symposium as it took place in the end, although an unusually large number of last minute changes beyond the planners' control compromised it to some extent. It proved helpful, however, in assuring the diversity and range which was in fact achieved.

The theory of partial differential equations, and especially of the physically relevant non-linear types, shows no signs of exhaustion as a primary source of important and stimulating problems which fascinate the mathematicians of applied interests. There are the basic questions revolving around the idea of what constitutes a well-formulated problem in this area which J. Leray touches in his paper and which have naturally led to the viewpoint, pioneered by Leray and others, of looking at solutions not individually but as families with topological and algebraic structure, capable of being supplemented by ideal elements, *etc.* There are the powerful function theoretical arguments which are being developed for the construction and analysis of important particular solutions, here represented by the methods with which P. Garabedian and his co-workers have recently attacked hydrodynamic flows possessing free surfaces. There are finally the fascinating problems posed by the nature of the singularities which can arise in the solution of these non-linear systems and the structure of the loci on which they can occur: A. H. Taub discusses such a problem using methods of differential geometry to analyze the configuration of interacting shock fronts in compressible flow. All of these are vital areas of substantial mathematical research today.

Ultimately destined, perhaps, to be of comparable importance in its mutually stimulating interaction with modern analysis is the area of signal and noise problems, illustrated by those presented in the paper of M. Kac. Here again the quest for analytic and measure-theoretical information on certain families of functions, containing individually observed time series as members, leads naturally to problems of substantial interest also to current front-line thinking in pure analysis. Another fertile field for mathematical ingenuity in applied contexts is furnished by what, in effect, are elementary problems but which are posed with

such a massive complexity of detail as to render the solution totally inaccessible to the traditional class-room approach. S. Chandrasekhar's discussion of eigenvalue problems of high order furnishes a good example how such an obstacle can be overcome by an ingenious mathematical trick, perhaps even, as in this case, without knowing in a rigorous way what accounts for the success of the trick.

Probably the least familiar aspect of the "new look" in applied mathematics is the steadily growing interaction between the combinatorial and algebraic mathematics of discrete structures on the one hand, and the use and operation of large-scale digital computers on the other. The use of Boolean algebra in the analysis of switching circuits is of course no longer new, but the idea of carrying forward basic research in Boolean algebras, as reported in the sequel by D. E. Muller, with the hope of anticipating the future needs of switching circuit design rather than being merely concerned with supplying the present ones, and to make use, moreover, of high-speed computing equipment in doing it—this gives evidence of the substantial development of a major new province of applied mathematics. In the same general area lies the work of Project SCAMP at the Institute for Numerical Analysis, whose purpose is that of exploring the use of high-speed automatic computers in the analysis of discrete structures. Some of it has been reported for the first time in any completeness at the Symposium by S. S. Cairns. Since summer after summer a sizable group of our ablest mathematicians participate in this work, it cannot fail in the long run to have a noticeable impact on American mathematical research.

Another important development, once more in the traditional vein of applied mathematics, is illustrated by the investigations, summarized by E. Montroll, concerning the energy distribution over the frequencies at which a molecular crystal lattice is capable of oscillating. It turns out that this analysis depends in an essential way on the relations known to exist between the topological structure of a closed manifold, here a certain kind of phase space, and the analytic properties of functions and integrals defined thereon. These problems are among the most intensively investigated ones in modern analysis, and if they should furnish the appropriate terms for analyzing significant aspects of current theories of molecular physics, we might be faced with their proliferous growth similar to the development of Hilbert space methods under the impact of quantum mechanics.

Everyone who scans this list will miss certain topics which he would consider as particularly interesting or important for the development of modern applied mathematics. Let him be assured that but few, if any, of these omissions were inadvertent. It is perhaps indicative of how much needs doing along these lines, that a first attempt could achieve as much and still leave so much untouched. Probably most glaring is the absence of a report, aimed at mathematicians, on the current problems and difficulties faced by quantum mechanical theories of fields. All endeavors of the Committee to provide such a report were unsuccess-

ful, and in hindsight this may have been all to the good, for the matter is serious enough to warrant a special effort.

3. Experimental Features. In arranging for a novel type of meeting, some innovations as regards organization are inevitable. The Symposium on Special Topics in Applied Mathematics introduced two such departures from standard practice. For one, the sessions of the Symposium were interspersed among the regular sessions of the Mathematical Society meeting, rather than grouped into a separate conference preceding or following the former and thus too easily avoided by the non-specialist. The example which was followed here is that of the American Physical Society whose meetings provide for a sizable number of invited half-hour papers, mixed in among the contributed papers either singly or in groups. Thus, the speakers were permitted some expository breadth in order to bring those of their colleagues not immediately concerned with the subject on hand up to date. The second innovation consisted in inviting representatives of other scientific fields to speak to mathematicians. The ones selected for this purpose naturally have extensive contact with mathematics in their daily scientific lives, yet they do not think of themselves as mathematicians. It is from such persons that a mathematician will generally get his most stimulating insight into the mathematical problems which arise in other scientific disciplines. Both of these experiments proved successful at the Symposium under discussion and their adoption, also in other than applied mathematical contexts, is recommended.

THE PHYSICAL FACTS AND THE DIFFERENTIAL EQUATIONS

JEAN LERAY, College de France

1. Introduction. A differential equation raises not only the problem of proving:

(A) the *existence* and *regularity* of its solutions; their uniqueness and their continuity with respect to the datum or at least their *compactness* for the datum running on a compact set.

Indeed, when those properties cannot be established, then the following alternatives arise:

(B) allowing the solution to have convenient singularities, to obtain *existence- and compactness-theorems without the regularity-theorem* (and indeed without the uniqueness-theorem even when uniqueness should be expected),

(C) to construct counter-examples proving that *the compactness-theorem does not hold.*

Assume that the equation is the conclusion of a physical theory expecting the existence, the regularity (and maybe the uniqueness) of its solutions. If (C) is possible, then the properties of the equation contradict the theory. On the contrary, if (A) holds, then the physical phenomenon is actually ruled by the equation. If not, and if (B) is possible, that is if the expected regularity (and uniqueness) is not obtained, then the physical theory neglected something which appears in the end to be not entirely negligible. The theory does not completely describe the phenomenon, which eludes perhaps any complete description. This often happens when the equation is non-linear, especially in the mechanics of fluids; their equations are based on the assumption that the flow is laminar, but the experiment shows turbulence.

Many papers have been devoted to problem (A); few to (B) and (C). The processes required for the establishment of (A) are often difficult; wherever (B) and (C) have been solved, it was by means of very simple processes, although at times somewhat concealed by technical details. We shall describe some of these here.

2. An equation which rules an evolution and for which (B) holds. Consider the Navier-Stokes system ruling incompressible viscous flows; it is parabolic and non-linear; the velocity is given at the initial time $t=0$ and has to be obtained for any $t>0$.

By successive approximations the flow can be calculated from $t=0$ to $t=T_1$, then from $t=T_1$ to $t=T_2$, *etc.* Let $T=\lim T_n$; the problem is solved and its solution is regular if $T=+\infty$. That occurs in some special cases: two-dimensional flows without boundary, sufficiently small data, *etc.* The proof is based on well known properties of the vortex; it can also be based on the relation expressing the conservation of the energy.

But the *homogeneity* of the Navier-Stokes system is such that the conservation of the energy and the smoothing action of the viscosity seem to be insufficient for maintaining the flow so regular that $T=+\infty$.

However, the experiment shows that the flow exists from $t=0$ to $t=+\infty$. Furthermore the Navier-Stokes system can be slightly modified in such a way that:

(a) the energy relation is preserved;

(b) the smoothing properties of the viscosity are so increased that the flow cannot become irregular and hence exists from $t=0$ to $t=+\infty$.

Let this modified system tend to the Navier-Stokes system; its solution converges weakly to one (or several) solutions of the Navier-Stokes system. This solution is not regular; it is *only square integrable* at any moment of time; it satisfies the Navier-Stokes system in the sense of L. Schwartz's distribution theory. Problem (B) is solved.

In some cases (three-dimensional flow without boundary, two-dimensional flow inside a convex boundary) a detailed study shows that such an irregular solution is regular except for a compact subset of values of t of measure zero.

The uniqueness of the flow and *the continuity* of its kinetic energy at the moments of time where it is not regular *are two open problems*; they are related.

No example of irregular flow has been obtained, but the above-mentioned homogeneity of the Navier-Stokes system suggests a process which probably leads to such an example.

3. An equation ruling an equilibrium and for which (A) holds in some cases, (C) in the other cases. Consider *the Dirichlet problem* for *non-linear, second order, elliptic* differential equations *in two independent variables*. The fixed point theory and the theory of the linear elliptic equations reduce the existence theorem to the compactness theorem, more precisely to the obtaining of *a priori* bounds for the solution and its first and second derivatives. These bounds follow from an extension of the maximum principle to the solutions of the studied equation, to their first and to their second derivatives. Assumptions about the equation and the Dirichlet data have to be made; they mean that the equation has a convenient behavior on the vertical cylinders (especially on the cylinder containing the graph of Dirichlet's data) or, if the equation belongs to Monge-Ampère's type, on the curves (especially on this graph).

The proof that those sufficient assumptions are necessary (when the equation behaves regularly on those cylinders and curves) is quite simple. It consists in the construction of a solution of the equation such that:

(a) this solution has at a point a vertical tangent plane or an infinite curvature;

(b) this solution is the limit of regular solutions defined on the same domain.

This is easily done by applying the Cauchy-Kowalewski theorem to the equation transformed by a change of coordinates or by a contact transformation.

The non-linear Dirichlet problem in three or more independent variables is

open. *A priori* bounds cannot be obtained again by a maximum principle; no existence or compactness theorems are known; problem (A) is not solved and probably problem (B) should be considered. However, the construction of the counter-examples showing that (C) holds could easily be extended.

Bibliography

Navier-Stokes system

Its origin, the linear Navier-Stokes system:

C. Oseen, Hydromechanik, (Leipzig, 1927).

Three-dimensional flow without boundary:

J. Leray, Acta math., vol. 63, 1934, pp. 193–248.

Two-dimensional flow inside a convex boundary:

J. Leray, Journal de math., vol. 13, 1934, pp. 331–418.

Regular flows:

J. Leray, Journal de math., vol. 12, 1933, ch. iv, pp. 64–82.
J. Leray, Comptes rendus Acad., vol. 194, 1932, pp. 1892.
W. Wolibner, Math. Zeits., vol. 37, 1933, pp. 698–726.
E. Hölder, Math. Zeits., vol. 37, 1933, pp. 727–738.
J. Leray et L. Robin, Comptes rendus Acad., vol. 205, 1937, p. 18.
P. Casal, Congrès intern. Mécan., 1952.

The general case:

E. Hopf, Math. Nachrichten, vol. 4, 1950, pp. 213–231.

Non-linear Dirichlet's problem

First studied by:

S. Bernstein, Math. Ann., vol. 69, 1910, pp. 82–136; vol. 95, 1926, pp. 585–594; vol. 96, 1927, pp. 633–647; Annales Ecole norm., vol. 27, 1910, pp. 233–256; vol. 29, 192, pp. 431–485.

His preliminaries about linear equations have been largely simplified by

J. Schauder, Math. Zeits., vol. 37, 1933, pp. 623–634; Studia Math., vol. 5, 1935, pp. 34–42.
L. Bers, L. Nirenberg (to be published)

The uniqueness assumption has been removed by

J. Leray, J. Schauder, Annales Ecole Norm. Sup., vol. 51, 1934.

H. Lewy and J. Schauder found a mistake in S. Bernstein's *a priori* bound of the first derivative of a quasi-linear equation (Math. Ann., vol. 69 p. 129: the expression of K is obviously wrong); furthermore, his conclusions are not invariant with respect to a change of coordinates; his study of the Monge-Ampère equation is incomplete. His results have been corrected and completed by

J. Leray, Journal de Math., vol. 17, 1938, pp. 89–104; vol. 18, 1939, pp. 249–284.

RECENT ADVANCES AT STANFORD IN THE APPLICATION OF CONFORMAL MAPPING TO HYDRODYNAMICS

P. R. GARABEDIAN, EDWARD McLEOD, JR., and MARTIN VITOUSEK,
Stanford University

The present paper forms an expository report of researches carried out recently at Stanford University by a group of mathematicians interested in free boundary problems of hydrodynamics. The discussion centers about a unified technique in geometric function theory.

The steady, irrotational motion of an incompressible, inviscid fluid in the z-plane is governed by an analytic complex potential $\zeta(z)=\phi+i\psi$. For a fluid of unit density, the pressure p is given by Bernoulli's equation

$$\tfrac{1}{2}\left|\zeta'(z)\right|^2+gy+p=\text{const.}, \tag{1}$$

where g is the gravitational constant. This law, in general, determines an additional boundary condition along free streamlines.

Following [3], we consider the uniform plane flow of a liquid past a small bubble of gas. We assume that the bubble is so small that the forces due to surface tension surpass significantly those due to gravity. Setting $g=0$ in (1) and resolving all forces along the normal to the free streamline, we obtain on the boundary C of the bubble a condition of the form

$$\left|\zeta'(z)\right|^2=T\kappa, \tag{2}$$

where κ is the curvature of the free boundary C and T represents the magnitude of the surface tension.

Denoting differentiation with respect to arc length s along C by a dot, we throw (2) into the form

$$\zeta'(z)^2\dot{z}=T\kappa\bar{z}, \tag{3}$$

and we obtain along C upon integration

$$q(z)=\int\zeta'(z)^2dz=iT\bar{z}, \tag{4}$$

where $q(z)$ is an analytic function in the flow region D. Hence $q(z)$ maps D conformally upon a Riemann surface bounded by a circle of radius T, while ζ maps D conformally upon the exterior of a horizontal slit. If $w(z)$ maps D upon the exterior of the unit circle, we can therefore write

$$\zeta=w+\frac{1}{w}, \tag{5}$$

$$q=Tw\frac{w^2+A^2}{1+A^2w^2}, \qquad A^2>1. \tag{6}$$

But since $dqdz = d\zeta^2$, we find

$$z = \frac{1}{T}\int \frac{\left(1-\frac{1}{w^2}\right)^2 dw}{\frac{d}{dw}\left[w\frac{w^2+A^2}{1+A^2w^2}\right]}, \tag{7}$$

where A is to be determined so that the numerator and the denominator of the integrand vanish simultaneously. We obtain the explicit formula

$$z = w - \frac{2}{3w} - \frac{1}{27w^3} \tag{8}$$

by integrating (7), and, together with (5), this determines completely the shape of the bubble and the corresponding flow.

In [4] a treatment is given of a free surface flow representing a water fountain in which gravity, rather than surface tension, is predominant. Taking $g=1/2$ and choosing suitable coordinates, we can put (1) in the form

$$\frac{dz}{d\zeta}\frac{d\bar{z}}{d\zeta} + \frac{1}{y} = 0 \tag{9}$$

along the free streamline. Following [2], we denote by $\lambda(\zeta)$ an analytic function in the flow region D which coincides with y on the free surface, and we substitute into (9) to obtain by analytic continuation the ordinary differential equation

$$z'(z' - 2i\lambda') + \frac{1}{\lambda} = 0, \tag{10}$$

where the prime indicates differentiation with respect to ζ. Along a vertical linear streamline, this differential equation in the unknown λ is real, and we can expect to obtain a real solution for real initial data.

The desired flow is required to rise along the y-axis until it reaches a free surface at the origin which spreads smoothly to either side and falls, joining two fixed vertical boundaries which descend to infinity. The continuation of the free streamline is linear and vertical, so that we can expect $\lambda(z)$ to map it onto the real axis. Since $\zeta(z)$ has the same mapping property, we are led to set $\lambda=\zeta$ in (10). Integration then gives

$$z = i\zeta - i\int_0^\zeta \sqrt{1+\frac{1}{\zeta}}\, d\zeta, \tag{11}$$

a formula which does, indeed, represent a gravity flow of the type described.

These techniques lead to further interesting free surface flows involving, respectively, surface tension and gravity. However, such examples do not exhaust the method. A successful treatment of cavitation in axially symmetric

flow yields to the present technique. The essential difficulty for this case stems from the more complicated partial differential equation

$$(12)\qquad \frac{\partial^2\psi}{\partial z\partial z^*}+\frac{1}{2(z-z^*)}\frac{\partial\psi}{\partial z}-\frac{1}{2(z-z^*)}\frac{\partial\psi}{\partial z^*}=0,\quad z=x+iy,\quad z^*=x-iy,$$

satisfied in the meridian plane by the stream function ψ governing the flow, but it can be largely overcome by appropriate exploitation of the Riemann function

$$(13)\qquad R(z,z^*;t,t^*)=\frac{(z-t^*)^{1/2}(t-z^*)^{1/2}}{t-t^*}F\left[\frac{(z-t)(z^*-t^*)}{(z-t^*)(z^*-t)}\right]$$

associated with this equation in the domain of two independent complex variables z and z^*. Here $F[w]=F(-1/2,\ -1/2,\ 1,\ w)$ is the hypergeometric series. Indeed, the stream function ψ of an axially symmetric free surface flow for which the analytic curve $\bar{z}=g(z)$ appears as the free boundary has an explicit representation

$$(14)\quad \psi(z,\bar{z})=\mathrm{Re}\left\{\frac{1}{2i}\int_{z_0}^{z}(z-g(t))^{1/2}(\bar{z}-t)^{1/2}F\left[\frac{(z-t)(\bar{z}-g(t))}{(z-g(t))(\bar{z}-t)}\right]g'(t)^{1/2}dt\right\},$$

where z_0 is a fixed point on the curve. Analysis of the conformal mapping properties of the analytic function $g(z)$ analogous to the study of $q(z)$ and $\lambda(z)$ carried through above yields explicit examples based on (14) of axially symmetric cavity flow in the large [1].

References

1. P. R. Garabedian, An Example of Axially Symmetric Flow with a Free Surface, Technical Report No. 5, Stanford University, 1952.

2. H. Lewy, On Steady Free Surface Flow in a Gravity Field, Technical Report No. 4, Stanford University, 1951.

3. Edward B. McLeod, Jr., An Application of the Schiffer Variation to the Free Boundary Problems of Hydrodynamics, Technical Report No. 15, Stanford University, 1953.

4. Martin Vitousek, Dissertation, Stanford University, to appear.

SINGULARITIES ON SHOCKS

A. H. TAUB, University of Illinois

1. Introduction. It is our purpose to discuss the situation which arises when a shock front is such that there are points on it at which the tangent to the shock, the curvature of the shock or some derivative of the curvature is discontinuous. Such shocks will be called shocks with singularities. They occur in many shock interaction problems; for example, the shock *MTRO*, in Figure 1, which illustrates the shock configuration arising in the Mach reflection of a plane shock from a rigid wall [1], may be considered as a shock with a singularity at the point T at which point the tangent to this shock is discontinuous.

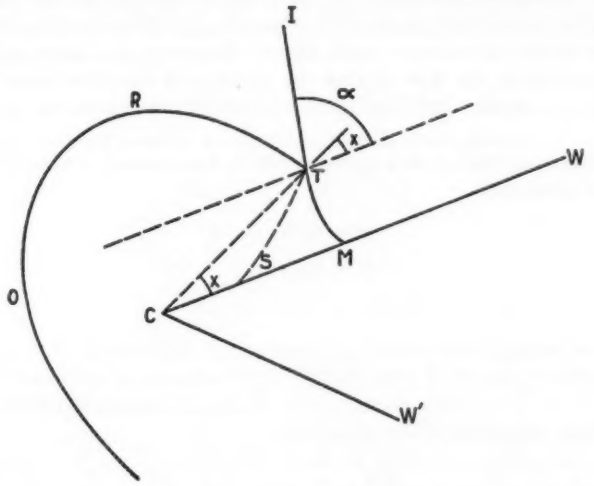


FIG. 1. Incident shock is TI; reflected shock is TRO; Mach shock is TM; slip stream is TS; wall is CW.

We shall restrict the discussion to two dimensional pseudo-stationary and two dimensional stationary flows. It will be sufficient to discuss in detail only one of the cases which will be chosen to be the first one, for the key equations needed in the discussion have been shown, in a previous publication [2], to be identical.

The shock configuration in a two dimensional pseudo-stationary flow at time t may be represented by the equations

$$x = ta_1(s)$$
$$y = ta_2(s), \tag{1.1}$$

where x and y are fixed Cartesian coordinates in the plane in which the shock

is moving, and s is a parameter along the shock which may be chosen so that

$$\left(\frac{da_1}{ds}\right)^2+\left(\frac{da_2}{ds}\right)^2=1 \tag{1.2}$$

at regular points. At a singular point of the shock some derivative of the functions $a_i(s)$ is discontinuous.

We shall assume that at time t each point of the region behind the shock with a singularity is occupied by a single fluid particle which has crossed the shock at some time t_0 earlier than t.

Let us denote by s_0 the value of s corresponding to a singular point of the shock and refer to the point on the shock as the point s_0. The locus of particles which have crossed the shock at s_0 at times $t_0<t$ must be a unique locus separating the region behind the shock with a singularity into just two regions in accordance with the assumption made above. However, this locus will be a locus of discontinuities in the flow behind the shock and therefore must be either a slip-stream or a shock. Only the former possibility is allowable as will now be shown.

The position at time t of a particle which has crossed a shock at a point s at time t_0 is given by

$$\begin{aligned} x &= ta_1(s,\tau) \\ y &= ta_2(s,\tau), \end{aligned} \tag{1.3}$$

where

$$\tau=\log\frac{t}{t_0} \tag{1.4}$$

and $a_i(s,\tau)$ are solutions of the equations

$$\frac{da_i}{d\tau}=U_i=u_i-a_i \tag{1.5}$$

where u_i is the particle velocity relative to the fixed Cartesian coordinate system; the $a_i(s,\tau)$ satisfy

$$a_i(s,0)=a_i(s), \tag{1.6}$$

where the functions on the right hand side of this equation are the functions entering into equations (1.1), the equations specifying the shock.

Equations (1.3) or the functions $a_i(s,\tau)$ for fixed s and variable t_0 give the locus of particles which have crossed the shock at a point s at times $t_0\leqq t$. We shall call these loci *streak-lines*. Our fundamental assumption implies that for a shock with a singularity at s_0 that the locus given by

$$\lim_{\epsilon\to 0} a_i(s_0+\epsilon,\tau)=a_i^+(\tau) \tag{1.7}$$

be identical with the locus given by

$$\lim_{\epsilon \to 0} a_i(s_0 - \epsilon, \tau) = a_i^-(\tau) \tag{1.8}$$

where τ is the parameter along this locus, defined by equation (1.4) with t fixed and t_0 variable. The two functions obtained by these two limiting procedures need not be identical in τ but they must represent the same curve. In Figure 1 this curve is represented by TS and called the *slip-stream*.

From equation (1.5) it follows that the tangent vector to the locus of particles that have crossed the shock at s at times $t_0 < t$ has components proportional to U_i. We denote by $U_i^+(a_1, a_2)$ the values of U_i at the point a_i on or above the singular locus described by (1.7) (in Figure 1 this region is STR) and by $U_i^-(a_1, a_2)$ the corresponding quantity at the point a_i on or below the singular locus described by (1.8) (in Figure 1 this region is STM). Since these two loci must be identical we must have

$$\epsilon_{ik} U_i^+(a_i^+(\tau)) U_k^-(a_i^+(\tau)) \equiv 0. \tag{1.9}$$

That is, their tangent vectors at the same points are equal:

$$\frac{U_i^+(a^+(\tau))}{v^+} = \frac{U_i^-(a^+(\tau))}{v^-} \tag{1.10}$$

where

$$(v^\pm)^2 = (U_1^\pm)^2 + (U_2^\pm)^2 \tag{1.11}$$

Equation (1.9) states that there is no flow across the singular locus (1.7) (or (1.8)) and hence this locus must be a slip-stream. It is a consequence of (1.9) and the Rankine-Hugoniot equations given below that

$$p^+(a_i^+(\tau)) \equiv p^-(a_i^+(\tau)) \tag{1.12}$$

where $p^+(a_i)$ is the pressure at a point a_i in the flow on or above the singular locus and $p^-(a_i)$ is the pressure at a point a_i in the flow on or below the singular locus. Equations (1.9) and (1.12) are identities in τ, and $a_i^+(\tau)$ represents a point on the singular locus.

In the subsequent sections of this paper we shall show how equations (1.10) and (1.12) and the derivatives of these equations may be used to determine geometric properties of the shock configuration at a singular point.

Since equations (1.9) and (1.12) are identities in τ, the derivatives of these equations with respect to τ must also hold. It follows from equation (1.9) that

$$\epsilon_{ik} U_{i,j}^+ U_j^+ U_k^- + \epsilon_{ik} U_i^+ U_{k,j}^- U_j^+ = 0,$$

where the comma denotes the ordinary derivative with respect to variable a and the argument of the functions U_i^+ and U_i^- is $a_i^+(\tau)$. In view of equation (1.10), this equation may be written as

$$\epsilon_{ik}\frac{U^+_{i,j}U^+_jU^+_k}{(v^+)^3}+\epsilon_{ik}\frac{U^-_iU^-_{k,j}U_j^-}{(v^-)^3}=0. \tag{1.13}$$

However, the curvature of a curve whose tangent vector is proportional to U_i is given by

$$K=-\frac{\epsilon_{ik}U_{i,j}U_jU_k}{v^3}. \tag{1.14}$$

Hence it follows from (1.13) that

$$K^+(a^+(\tau))\equiv K^-(a^+(\tau)), \tag{1.15}$$

where K^+ is the curvature of the locus described by (1.7) and K^- is the curvature of the locus described by (1.8). Differentiating both sides of equation (1.15) we obtain

$$K^{+\prime}=K^{-\prime} \tag{1.16}$$

where the prime denotes the derivative with respect to the arc length along the singular locus. Similarly we may show that

$$K^{+(n)}=K^{-(n)} \tag{1.17}$$

where the superscript n indicates the nth derivative with respect to the arc length of the singular locus.

We may also show that equation (1.12) implies that

$$p^{+(n)}=p^{-(n)}. \tag{1.18}$$

Equations (1.17) and (1.10) are merely the formal consequences of the requirement that equations (1.7) and (1.8) represent the same locus, for the locus is intrinsically specified in terms of the tangent vector, the curvature, and the derivatives of the curvature.

2. The equations describing the flow. In order to determine the geometric properties of a shock at a singular point we use equations (1.17), (1.10), and (1.18) evaluated at the shock (*i.e.*, at $\tau=0$) and the relations obtained by Thomas [3] relating the curvature and the derivatives of the curvature with respect to arc length of a streak-line in stationary flow with the corresponding quantities of the shock from which the streak-line emanates. The identical relations may be used in pseudo-stationary flows as follows from the results of Taub [2]. Thus we assume analyticity of the flow in regions behind the shock front with a singularity and replace equations (1.9) and (1.12) by sequences of conditions at the singular point of the shock, namely, by equations (1.10), (1.17) and (1.18) evaluated at $\tau=0$. These equations are then replaced by conditions involving the quantities characterizing the shock by using results from consideration involving the differential equations that must be satisfied by the

flow variables and the Rankine-Hugoniot equations that must hold across a shock.

For pseudo-stationary flows the former equations may be written as: the conservation of mass,

$$(\rho U_i)_{,i} + 2\rho = 0, \tag{2.1}$$

the conservation of momentum,

$$U_j U_{i,j} + U_i + \frac{1}{\rho} p_{,i} = 0, \tag{2.2}$$

and the conservation of energy,

$$U_i U_j U_{i,j} + v^2 - c^2(U_{i,i} + 2) = 0, \tag{2.3}$$

where ρ is the density and

$$c^2 = \frac{\gamma p}{\rho}. \tag{2.4}$$

The Rankine-Hugoniot equations may be written as

$$U_t = U_i\lambda_i = U_{1i}\lambda_i = U_{1t} \tag{2.5}$$

$$y - 1 = \frac{2\gamma}{\gamma + 1}(\sigma^2 - 1) \tag{2.6}$$

$$\eta = \frac{(\gamma + 1)\sigma^2}{2 + (\gamma - 1)\sigma^2} \tag{2.7}$$

$$U_n = v_i\epsilon_{ij}\lambda_j = \frac{c_1}{(\gamma + 1)\sigma}((\gamma - 1)\sigma^2 + 2), \tag{2.8}$$

where λ_i are the components of the tangent vector to the shock,

$$\sigma = \frac{U_{1i}\epsilon_{ij}\lambda_j}{c_1} = \frac{U_{1n}}{c_1} \tag{2.9}$$

$$y = \frac{p}{p_1} \tag{2.10}$$

$$\eta = \frac{\rho}{\rho_1} \tag{2.11}$$

and we have used the subscript 1 to denote quantities evaluated on one side of the shock, a side which we will assume to be uniform. Thus in Figure 1 the subscript 1 may refer to the region TRI or to the region to the right of TM.

In the work of Thomas referred to above, equations (2.1) to (2.3) and the derivatives of equations (2.5) to (2.8) along the shock when evaluated at the

shock were considered as algebraic equations for the quantities $U_{i,j}$, $p_{,i}$ and $\rho_{,i}$. The solutions of these equations were then used to give explicit expressions for the derivatives of the flow variables behind a shock in terms of properties of the shock. Similar methods were used to obtain relations between higher derivatives of the quantities U_i, p and ρ at a point behind a shock in terms of the derivatives of the equations characterizing the shock.

3. Thomas' results. We shall describe a method for obtaining Thomas' results and the addition equations needed for the carrying out of the procedure outlined above. We consider the functions

$$a_i = a_i(s, \tau) \tag{3.1}$$

which are solutions of equations (1.5) subject to the initial conditions (1.6) as a transformation from the variables a_i to a new set of independent variables s and τ. Then

$$\frac{\partial a_i}{\partial \tau} = U_i \tag{3.2}$$

$$\frac{\partial a_i}{\partial s} = \lambda_i, \tag{3.3}$$

where λ_i is defined by equations (3.3) and is such that

$$\frac{\partial \lambda_i}{\partial \tau} = \frac{\partial U_i}{\partial s}. \tag{3.4}$$

It then follows that

$$\frac{\partial \tau}{\partial a_i} = \frac{\epsilon_{ij}\lambda_j}{U_n} \tag{3.5}$$

and

$$\frac{\partial s}{\partial a_i} = -\frac{\epsilon_{ij}U_j}{U_n}, \tag{3.6}$$

where

$$U_n = \epsilon_{ij}U_i\lambda_j. \tag{3.7}$$

We now have

$$\begin{aligned} U_{i,j} &= \frac{\partial U_i}{\partial \tau}\frac{\partial \tau}{\partial a_j} + \frac{\partial U_i}{\partial s}\frac{\partial s}{\partial a_j} \\ &= \frac{\partial U_i}{\partial \tau}\frac{\epsilon_{jk}\lambda_k}{U_n} - \frac{\partial \lambda_i}{\partial \tau}\frac{\epsilon_{jk}U_k}{U_n}. \end{aligned} \tag{3.8}$$

Hence

$$U_{i,i} = \frac{\partial \log U_n}{\partial \tau} \tag{3.9}$$

and

$$\epsilon_{ij} U_{i,j} = \frac{1}{U_n}\left(2\,\frac{\partial U_i}{\partial s}\,U_i - \frac{\partial U_i}{\partial \tau}\right) \tag{3.10}$$

where

$$U_i = U_i \lambda_i. \tag{3.11}$$

Equations (3.9) enable us to obtain a first integral of the conservation of mass equation. Equation (2.1) may be written as

$$\frac{\partial \rho}{\partial \tau} + \rho U_{i,i} + 2\rho = 0$$

or as

$$\frac{\partial \log \rho}{\partial \tau} + \frac{\partial \log U_n}{\partial \tau} + 2 = 0.$$

Hence we must have

$$\rho U_n = \rho_0 U_{0n} e^{-2\tau}, \tag{3.12}$$

where the arguments of the functions on the left hand side of this equation are s and τ and we have introduced the convention that

$$f_0 = f(s, 0). \tag{3.13}$$

The conservation of momentum equations may be written as

$$\frac{\partial U_i}{\partial \tau} + U_i = -\frac{1}{\rho}\,p_{,i}. \tag{3.14}$$

Multiplying these equations by U_i and summing we have

$$\frac{\partial U_i}{\partial \tau}\,U_i + v^2 = -\frac{1}{\rho}\,\frac{\partial p}{\partial \tau} \tag{3.15}$$

or

$$\frac{1}{2}\,e^{-2\tau}\frac{\partial}{\partial \tau}\left(e^{2\tau} v^2\right) = -\frac{1}{\rho}\,\frac{\partial p}{\partial \tau}, \tag{3.16}$$

and multiplying by λ_i and summing we have

$$\frac{\partial U_t}{\partial \tau} + U_t = -\frac{1}{\rho}\frac{\partial p}{\partial s} + \frac{1}{2}\frac{\partial v^2}{\partial s}. \tag{3.17}$$

The conservation of energy equation may be written as

$$\frac{\partial}{\partial \tau}(p\rho^{-\gamma}) = 0$$

and admits the first integral

$$p = p_0\left(\frac{\rho}{\rho_0}\right)^{\gamma}. \tag{3.18}$$

Hence in the s, τ coordinate system, in which the curves $s =$ constant are the loci of particles which have crossed the shock at earlier times, the streak-lines, and the curves $\tau =$ constant are a family of curves which include the shock (for the value $\tau = 0$), the conservation equations admit two first integrals and take on the relatively simple form (3.16) and (3.17) in addition. The use of these coordinates in the integration of the equations of pseudo-stationary flows will be discussed elsewhere.

Equation (2.3) may be written as

$$U_i\frac{\partial U_i}{\partial \tau} - \frac{c^2}{U_n}\frac{\partial U_n}{\partial \tau} = 2c^2 - U_iU_i.$$

If we let

$$\lambda^2 = \lambda_i\lambda_i,$$

then

$$\lambda^2 v^2 = \lambda^2 U_iU_i = U_n^2 + U_t^2$$

$$\lambda^2 U_i\frac{\partial U_i}{\partial \tau} + \lambda_i\frac{\partial \lambda_i}{\partial \tau}U_jU_j = U_n\frac{\partial U_n}{\partial \tau} + U_t\frac{\partial U_t}{\partial \tau}.$$

Hence

$$U_i\frac{\partial U_i}{\partial \tau} = \frac{1}{\lambda^2}\left(U_n\frac{\partial U_n}{\partial \tau} + U_t\frac{\partial U_t}{\partial \tau}\right) - \frac{v^2}{\lambda^2}\lambda_i\frac{\partial U_i}{\partial s}$$

and

$$U_n\frac{\partial U_n}{\partial \tau} + U_t\frac{\partial U_t}{\partial \tau} - v^2\lambda_i\frac{\partial U_i}{\partial s} - \frac{c^2\lambda^2}{U_n}\frac{\partial U_n}{\partial \tau} = 2c^2\lambda^2 - v^2\lambda^2.$$

Substituting in this equation from equation (3.17) we obtain

$$\frac{\partial U_n}{\partial \tau} = \frac{U_n}{U_n^2 - c^2\lambda^2}\left(2c^2\lambda^2 - U_n^2 + v^2\lambda_i\frac{\partial U_i}{\partial s} + U_t\left(\frac{1}{\rho}\frac{\partial p}{\partial s} - \frac{1}{2}\frac{\partial v^2}{\partial s}\right)\right)$$

or

$$\frac{\partial U_n}{\partial \tau} = \frac{U_n}{U_n^2 - c^2\lambda^2}\left[2(c^2\lambda^2 - U_n^2) + v^2\left(\lambda^2 + \lambda_i \frac{\partial U_i}{\partial s}\right) + U_i\left(\frac{1}{\rho}\frac{\partial p}{\partial s} - v\frac{\partial v}{\partial s} - U_i\right)\right]. \tag{3.20}$$

At a point on the shock, equations (3.17) and (3.20) express the rate of change of the pseudo-velocity vector U along the streak-lines in terms of the derivatives of the flow variables along the shock. That is, the right hand sides of these equations may be computed from the functions a_i occurring in equations (1.1) and from equations obtained by differentiating the Rankine-Hugoniot equations (2.5) to (2.8). We may also compute $\partial p/\partial \tau$ and $\partial \rho/\partial \tau$ behind a shock from the right hand sides of equations (3.17) and (3.20), that is, in terms of the derivatives of the flow variables along a shock.

Thus

$$\frac{\partial \rho}{\partial \tau} = -2\rho - \frac{\rho}{U_n}\frac{\partial U_n}{\partial \tau}.$$

Substituting from (3.20) we obtain

$$\frac{\partial \rho}{\partial \tau} = -\frac{\rho}{U_n^2 - c^2\lambda^2}\left[v^2\left(\lambda^2 + \lambda_i \frac{\partial U_i}{\partial s}\right) + U_i\left(\frac{1}{\rho}\frac{\partial p}{\partial s} - v\frac{\partial v}{\partial s} - U_i\right)\right]. \tag{3.21}$$

It follows from the equation preceding equation (3.18) that

$$\frac{\partial p}{\partial \tau} = c^2\frac{\partial \rho}{\partial \tau} = \frac{-\rho c^2}{U_n^2 - c^2\lambda^2}\left[v^2\left(\lambda^2 + \lambda_i \frac{\partial U_i}{\partial s}\right) + U_i\left(\frac{1}{\rho}\frac{\partial p}{\partial s} - v\frac{\partial v}{\partial s} - U_i\right)\right]. \tag{3.22}$$

The curvature of a streak-line may also be expressed in terms of the flow variables and their derivatives with respect to s. From the definition of curvature we have

$$-v^3K = \epsilon_{ik}\frac{\partial U_i}{\partial \tau}U_k = -\frac{1}{\rho}\epsilon_{ik}p_{,i}U_k$$

as follows from the conservation of momentum equations. However

$$p_{,i} = \frac{1}{U_n}\left(\frac{\partial p}{\partial \tau}\epsilon_{il}\lambda_l - \frac{\partial p}{\partial s}\epsilon_{il}U_l\right).$$

Hence

$$v^3K = \frac{1}{\rho U_n}\left(\frac{\partial p}{\partial \tau}U_i - \frac{\partial p}{\partial s}v^3\right), \tag{3.23}$$

where we may substitute from the equation (3.22) for $\partial p/\partial \tau$.

Equations (3.22) and (3.23) when evaluated for $\tau = 0$, that is, at the shock, give explicit expressions for $p^{\pm(1)}$ and $K^{\pm(0)}$ needed in the determination of the geometric properties of a shock with a singularity by means of equations (1.17) and (1.18). Explicit expressions for $p^{\pm(n)}$ and $K^{\pm(n)}$ may also be obtained from equations (3.22) and (3.23) respectively as follows: Either of these equations may be differentiated n times with respect to τ. The right hand sides may be reduced to expressions involving only derivatives with respect to s. The derivatives with respect to s may be evaluated by differentiating the Rankine-Hugoniot equations.

We shall demonstrate the method outlined by evaluating equations (3.22) and (3.23) for $\tau = 0$. It follows from equations (2.5) to (2.8) that if the flow ahead of the shock is uniform, then (*cf.* [2] eq. (4.2) to (4.6))

$$\begin{aligned} \frac{\partial U_{1i}}{\partial s} &= -\lambda_i \\ \frac{\partial U_{1n}}{\partial s} &= kU_{1t} = c_1\frac{\partial \sigma}{\partial s} \\ \frac{\partial U_{1t}}{\partial s} &= -1 - kU_{1n} = -1 - kc_1\sigma = \frac{\partial U_t}{\partial s} \end{aligned} \tag{3.24}$$

where k is the curvature of the shock and λ_i in these equations represents the tangent vector to the shock. From the definition of s in equations (1.1) it follows that

$$\lambda^2(s, 0) = \lambda_i\lambda_i = 1. \tag{3.25}$$

Further it may be shown that at $\tau = 0$

$$\begin{aligned} \frac{\partial p}{\partial s} &= \frac{4kU_tU_n\rho}{\gamma+1} \\ \frac{\partial \rho}{\partial s} &= \frac{4kU_t\rho_1}{(\gamma+1)U_n}(2-(\gamma-1)(\eta-1)) = \rho_1\frac{\partial \eta}{\partial s} \\ \frac{\partial U_n}{\partial s} &= kU_t\left(\frac{2(\gamma-1)}{\gamma+1} - \frac{1}{\eta}\right) \end{aligned} \tag{3.26}$$

and

$$1 + \lambda_i\frac{\partial U_i}{\partial s} = -k(\eta-1)U_n \tag{3.27a}$$

$$U_t + v\frac{\partial v}{\partial s} = -kU_tU_n\left(\eta + \frac{1}{\eta} - \frac{2(\gamma-1)}{\gamma+1}\right). \tag{3.27b}$$

Substituting from these equations into (3.22) and (3.23) we obtain at $\tau = 0$

$$\frac{\partial p}{\partial \tau} = -\frac{\rho c^2 k U_n}{U_n^2 - c^2}\left(\left(\frac{8}{\gamma+1} + \frac{1-\eta}{\eta}\right)U_t^2 - (\eta-1)U_n^2\right) \tag{3.28}$$

and

$$K = k\frac{U_t}{v^3}\left[\frac{c^2}{U_n^2 - c^2}\left(\frac{8}{\gamma+1} + \frac{1-\eta}{\eta}\right)U_t^2 - (\eta-1)U_n^2\right] \tag{3.29}$$

respectively.

It has been shown by Thomas [3] for stationary flows and by Taub [2] for pseudo-stationary flows that at $\tau = 0$

$$K^{(n)} = G_n(\eta, U_t/U_n)k^{(n)} + H_n \tag{3.30}$$

where $K^{(n)}$ is the quantity defined in section 1, $k^{(n)}$ is the nth derivative of the curvature of the shock with respect to s, and H_n is a polynomial in k and its derivatives with respect to s of order less than n.

It follows from the arguments given above that at $\tau = 0$

$$p^{(n)} = f_n(\eta, U_t/U_n, k, k^{(1)}, \cdots, k^{(n-1)}). \tag{3.31}$$

That is, behind the shock $p^{(n)}$ is a function of the strength of the shock, its inclination with respect to the flow incident upon it, the curvature of the shock and the first $n-1$ derivatives of this curvature with respect to s.

4. Conditions at singular points of a shock. The formulas of the preceding section enable us to translate the conditions which must obtain at a singular point of a shock, namely, equations (1.10), (1.17) and (1.18), into sequences of algebraic relations between the variables $\eta^{\pm}$, $U_t^{\pm}/U_n^{\pm}$, $k^{\pm}$ and $k^{\pm(i)}$ $(i = 1, 2, \cdots)$.

A shock which has a straight portion and a singular point at which the curvature or some derivative of the curvature jumps to a non-vanishing value cannot have a uniform flow behind the straight portion of the shock at the singular point. This follows from the fact that if the flow were uniform, equations (3.30) would apply for the calculation of $K^{+(n)}$ and hence these quantities would vanish. However, the quantities $K^{-(n)}$ would not vanish and it would be impossible to satisfy equations (1.17). The discussion of a special case of a shock with such a singularity is contained in [2], where a shock with a point at which the curvature jumps from zero to a non-vanishing value is discussed.

In a shock configuration such as shown in Figure 1, a shock with uniform flows incident on it and with a singularity at which the tangent is discontinuous may exist. The sequence of algebraic conditions which must hold may then be paired as follows:

$$
\begin{aligned}
v^- U_i^+ &= v^+ U_i^- \\
p^+ &= p^- ,
\end{aligned}
\tag{4.1}
$$

the first of these being (1.10) and the second being (1.18) for $n=0$;

$$
\begin{aligned}
K^+ &= K^- \\
p^{+(1)} &= p^{-(1)}
\end{aligned}
\tag{4.2}
$$

where we have used (1.17) and (1.18); and the remaining equations may be written as

$$
\begin{aligned}
K^{+(n)} &= K^{-(n)} \\
p^{+(n+1)} &= p^{-(n+1)} \qquad\qquad n = 1, 2, \cdots .
\end{aligned}
\tag{4.3}
$$

Equations (4.1) are the fundamental equations of the three-shock theory (*cf.* [1]). They may be solved for η^+, U_j^+/U_n^+, η^- and U_j^-/U_n^- as functions of the strength and angle between the flow incident on and the normal to the shock TI of Figure 1. The solutions are not unique and only very special ones exist for some values of these two parameters characterizing the shock TI.

Equations (4.1) and (4.2) have been solved by Clutterham and Taub [4]. Because the function H_0 of (3.30) vanishes and because $p^{(1)}$ depends linearly on k (*cf.* (3.28)) these equations are independent of $k^\pm$ and imply a relation between the two variables characterizing the shock TI. This in turn determines possible world lines of the point T in Figure 1. However, if either k^+ or k^- is given, then the remaining curvatures, k^- or k^+ and K^+ and K^-, are determined.

The sequence of equations (4.3) may be used to determine $k^{\pm(n)}$ and hence $K^{\pm(n)}$ as functions of k^+ or k^-.

The fact that k^- cannot be determined from the equations (4.1), (4.2) and (4.3) is to be expected; for in a shock configuration such as in Figure 1, the shock TM must satisfy an additional condition, namely, it must be orthogonal to the wall CW. Such a requirement serves to determine k^- (*cf.* [4]).

Thus a shock configuration such as that occurring in Figure 1 may be determined to an arbitrary degree of accuracy by solving equations (4.1), (4.2) and (4.3) with n in the latter equations chosen sufficiently great and satisfying the boundary condition on CW. However, the solution will not be unique for a given shock TI.

References

1. W. Bleakney and A. H. Taub, Interaction of shock waves, Rev. Mod. Phys., vol. 21, 1948, pp. 584–605.

2. A. H. Taub, Curved shocks in pseudo-stationary flows, Ann. Math., vol. 58, no. 3, 1953, pp. 501–527.

3. T. Y. Thomas, The distribution of singular shock directions, J. Math. and Phys., vol. 28, no. 3, 1949, pp. 153–172.

4. D. R. Clutterham and A. H. Taub, Numerical results on the shock configuration in Mach reflection, to appear in the Proceedings of the Sixth Symposium of Applied Mathematics.

SIGNAL AND NOISE PROBLEMS

M. KAC, Cornell University

In this brief report I shall limit myself to a cursory discussion of several problems chosen on grounds of personal predilection and with the view of showing how "pure" and "applied" mathematics can intermingle with profit to both.

1. The problem of detection. This is a purely statistical problem which can be formulated as follows: Let $x(t)$ be a stationary random process (noise) and $s(t)$ a signal. An observer receives a record $y(t)$ which is either $x(t)$ or $x(t)+s(t)$ and is to decide whether the signal is present or not.

In the simplest case, when $s(t)$ is periodic with period θ and a finite number of observations is made at times t_1, $t_1+\theta$, $\cdots$, $t_1+(n-1)\theta$ and if furthermore θ is long compared with the correlation time of the noise (*i.e.*, $x(t_1+j\theta)$, $j=1, 2, \cdots, n-1$, are independent), the problem can be solved by the use of the Neyman-Pearson theory. A detailed account can be found in [1]. More recently various authors [2] considered the sequential approach to the problem. In its general form the problem clearly belongs to decision theory and it may be hoped that here the theories of the late A. Wald will play an important part.

2. Spectra and correlation. The relation between the power spectrum $A(\omega)$ and the correlation function $\rho(t)$,

$$\rho(t) = \int_{-\infty}^{\infty} A(\omega) \cos \omega t \, d\omega, \tag{2.1}$$

is a standard tool in the theory of noise. It is often desirable to consider processes for which

$$\rho(t) = 0 \qquad |t| > A. \tag{2.2}$$

The relations (2.1) and (2.2) combined with the fact that $A(\omega) \geqq 0$ induce strong restrictions on $\rho(t)$. One can show, for instance, [3] that if n is an integer

$$\left|\rho\left(\frac{A}{n}\right)\right| \leqq \cos\frac{\pi}{n+1}\rho(0), \tag{2.3}$$

the constant being the best possible.

3. Integral equations. In the theory of radio receivers with square-law detectors one is led to the problem of finding the distribution function of an expression

$$\int_0^{\infty} K(\tau)\{x^2(t-\tau) + y^2(t-\tau)\}\,dt, \tag{3.1}$$

where $x(t)$ and $y(t)$ are independent, stationary Gaussian processes with the same correlation function $\rho(t)$.

The characteristic function of the distribution function of (3.1) is given by (see [4])

$$D^{-1}(i\xi),$$

where $D(\lambda)$ is the Fredholm determinant of the integral equation

$$\lambda \int_0^\infty \rho(s-t)K(t)\phi(t)dt = \phi(s). \tag{3.2}$$

Integral equations of the form (3.1) appear in other branches of pure and applied mathematics and it is at least amusing to contemplate solving them by building a corresponding receiver and determining the distribution function experimentally.

4. Zeros of random functions. This is an extremely interesting and difficult problem where much further work needs to be done. If $x(t)$ is a stationary Gaussian process with power spectrum $A(\omega)$, the average number of zeros per unit time is given by Rice's formula [5]:

$$2\left\{\frac{\int_{-\infty}^{\infty}\omega^2 A(\omega)d\omega}{\int_{-\infty}^{\infty}A(\omega)d\omega}\right\}^{1/2}. \tag{4.1}$$

The fact that by counting zeros one can obtain information about the spectrum is in itself of great practical interest. It has been, for instance, applied to turbulence by H. W. Liepmann and his group at the California Institute of Technology. A more detailed study of the distribution of zeros of random functions for even the simplest processes encounters great analytical difficulties. A closely related problem is the following:
Let

$$f(t) = \sum_1^n a_k \cos 2\pi(\lambda_k t + \phi_k) \tag{4.2}$$

and assume that the frequencies λ_k are rationally independent. Let $N(T, a)$ be the number of roots of

$$f(t) = a$$

in $0 \leqq t \leqq T$.

It can then be demonstrated [6] that

$$\lim_{T\to\infty}\frac{N(T,a)}{T} = \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\int\frac{\cos a\xi}{\eta^2}\left\{\prod_{k=1}^{n}J_0(a_k\eta) - \prod_{k=1}^{n}J_0(a_k\sqrt{\xi^2+\lambda_k\eta^2})\right\}d\xi d\eta. \tag{4.3}$$

This is the counterpart of (4.1) and can be interpreted by saying that the average distance between consecutive a-values of $f(t)$ is given by the inverse of the expression on the righthand side of (4.3). If one asks for the average of the *square* of the distance between consecutive a-values one runs into difficulties which, at least at present, appear insurmountable. This is already true for the simplest case

$$f(t) = a_1 \cos \lambda_1 t + a_2 \cos \lambda_2 t$$

(except for special values of a_1, a_2 and a). An interesting application of (4.3) to the theory of unimolecular reaction rates was made recently by N. B. Slater [7]. In conclusion let me mention another related problem, this time with no practical implications.

Consider a polynomial of degree n

$$\sum_{k=0}^{n} X_k t^k \tag{4.4}$$

whose coefficients are independent, normally distributed, random variables each having mean 0 and variance 1. It is then easy to show [8] that the average number of real roots of (4.3) is asymptotically

$$\frac{2}{\pi} \log n. \tag{4.5}$$

Moreover, a tedious but rather elementary calculation shows that the standard deviation about the mean is of lower order and consequently it is very rarely that a random algebraic equation of high degree will have a number of real roots which is significantly different from (4.5).

These conclusions remain valid for a much wider class of independent random variables [9] but proofs become enormously more tedious. For the simplest case

$$\text{Prob.}\,\{X_k = 1\} = \text{Prob.}\,\{X_k = -1\} = \tfrac{1}{2}$$

the proof that (4.5) is still asymptotically the average number of real roots is lacking!

The present exposition was naturally permeated with probabilistic considerations. But perhaps it is not too idle and inappropriate to contemplate here the possibility of a statistical approach to various questions in pure mathematics. As an example let me consider the following question: how good is the classical Descartes' rule of signs?

As applied to an individual algebraic equation the question is largely meaningless. Interpreted statistically it can be properly formulated and answered. The average number of changes of sign in (4.4) is $n/2$ (this is to be compared with $\pi^{-1} \log n$ which is the average number of positive real roots of (4.4)).

However, one can do better. If one considers the polynomial

$$\left(\sum_0^n t^k\right)\left(\sum_0^n X_k t^k\right),$$

the number of real roots is the same as for (4.4); the average number of changes of sign can now be shown to be of the order $C\sqrt{n}$. This is about the best one can do, and yet $C\sqrt{n}$ is still so far from the correct order $\pi^{-1}\log n$ that we must conclude that Descartes' rule of sign is extremely unlikely to give a good estimate for equations of high degree.

Bibliography

1. J. L. Lawson and G. E. Uhlenbeck, Threshold Signals, McGraw-Hill, 1950.

2. See *e.g.*, W. C. Fox, Signal detectability, Eng. Rev. Inst., Univ. of Michigan. Here references to previous work, especially that of D. Middleton, are given.

3. R. P. Boas, Jr. and M. Kac, Inequalities for Fourier transforms of positive functions. Duke Math. J., vol. 12, 1945, pp. 189–206. (See also the errata *ibid.* 15, 1948, pp. 105–109.)

4. M. Kac and A. J. F. Siegert, On the theory of noise in radio receivers with square law rectifiers. J. Appl. Phys., vol. 18, 1947, pp. 383–397.

5. S. O. Rice, Mathematical analysis of random noise. Bell. Tech. J., vol. 23, 1944, pp. 282–332.

6. M. Kac, On the distribution of values of trigonometric polynomials with linearly independent frequencies. Am. J. Math., vol. 65, 1943, pp. 609–615. See also a recent paper by N. B. Slater: Some formulas of P. Stein and others concerning trigonometrical sums. Proc. Cam. Phil. Soc., vol. 50, 1950, pp. 33–39.

7. N. B. Slater, Aspects of a theory of unimolecular reaction rates. Proc. Roy. Soc. A194, 1948, pp. 112–131.

8. M. Kac, On the average number of real roots of random algebraic equations. Bull. Amer. Math. Soc., vol. 49, 1943, pp. 314–320.

9. M. Kac, On the average number of real roots of random algebraic equations II, Proc. London Math. Soc., vol. 50, 1949, pp. 390–408.

BOOLEAN ALGEBRAS IN ELECTRIC CIRCUIT DESIGN

D. E. MULLER, University of Illinois

A brief description of how Boolean algebras may be used in the design of computer circuits is followed by an account of studies in the field of Boolean algebras which were carried out with the aid of the ILLIAC computer at the University of Illinois. The computer was used to investigate relationships between functions in Boolean algebras which uncovered symmetries leading to the development of certain theorems. These theorems turned out on later expansion to have a number of applications to computer design. Hence, the work under discussion is quite distinct from the customary use of the computer in carrying out massive algebraic manipulations with the objective of settling some specific design problems.

One of the main motives for the introduction of algebraic techniques into the design of switching circuits is the problem of devising the simplest possible circuit toward a specified end under a given criterion of simplicity. In principle this can, of course, always be achieved by brute force since it can be reduced to the trial of a finite, although generally very large number of combinations. A mathematically satisfying solution, however, has not yet been found and will certainly require a deeper insight into the basic structure of Boolean functions than available at present.

The investigations are concerned with this latter type of functions and therefore apply only to circuits of the so-called static type in which the outputs are expressible as Boolean functions of the inputs. This excludes, in particular, digital circuits with feed-back loops. Any Boolean function of p independent variables can be expanded into a canonical representation consisting of a linear expression in 2^p standard functions with coefficients which are either zero or one. The set of coefficients serves to completely define the function being represented, and conversely any ordered set of 2^p zeros and ones represents a Boolean function. Hence every such function can be regarded as corresponding to a vertex of the 2^p-dimensional unit cube, and the geometric properties of the latter should be reflected in algebraic properties of the corresponding totality of Boolean functions. A particular orthogonal coordinate system, with its origin at the center of the cube, is introduced by passing axes through the vertices corresponding to the independent variables themselves and through the points corresponding to all functions which can be obtained from the latter, using only the operation "exclusive or."

Coordinates of points with respect to these axes for the case $p=4$ were studied by the computer. For each point the number of coordinates of absolute magnitudes 0, 1, 2, $\cdots$, 8 were obtained, and all points for which these nine numbers agreed were put into the same equivalence class. Such equivalence classes represent sets of functions bearing similar relationships to the axes and therefore, presumably, similar algebraic form. There were 8 such classes of

which 3 attracted particular attention, because the number of functions contained in them was 2048, an exact power of two. Jointly they contained all points corresponding to functions, each one of which can be expressed as a sum of products of no more than two of the independent variables, and it turned out that the squared distance between any pair of such points is never less than 4. This suggested giving a special name to the set of all points corresponding to functions which are the sum of products of no more than $p-q$ of the independent variables: this is a *net of order* q. It can then be proved that the distance between any pair of points in a net of order q is at least 2^q.

An unexpected application of this concept was found in the field of error detection which at the same time suggested a number of problems. In this case one wishes to find the most numerous set of vertices of an n-dimensional cube, any pair of which is separated by at least the distance d. The question whether for $n=2^p$, $d=2^q$, the net of order q furnishes a solution has not been completely answered.

The simpler question of whether additional vertices can be found such that the distance property remains valid for the augmented set has been answered affirmatively for large enough p except when $q=0, 1, p-1$, and p. For these latter values of q, the net cannot be augmented for any value of p. The addition of further points is known to be possible when $p=7$, $q=4$. For $p=5$, $q=3$, a computer study has shown that no augmentation of the net is possible. The case $t=6$ remains open.

COMPUTATIONAL ATTACKS ON DISCRETE PROBLEMS

S. S. CAIRNS, University of Illinois

This is a summary of some recent work done at the University of California at Los Angeles under the so-called SCAMP project, of which I was chairman during the summers of 1952 and 1953. This project represents the first organized effort to attack discrete problems with the aid of large-scale computing equipment. It is a pioneer effort and only in the early stages of its expected development. The objectives of the computational procedures are (1) to obtain numerical information useful in the development of general theories, (2) to verify hypotheses which are beyond the reach of hand computations, and (3) to accumulate data on the basis of which to make conjectures as foundations for further research.

Thus far much, and perhaps too much, attention has been devoted to the existence problem for finite projective planes of order m. This can be interpreted as the problem of finding the incidence matrix A of such a plane; that is, a matrix A of order $n=m^2+m+1$, whose elements are all zeros or ones, there being exactly $m+1$ ones in each row and exactly one column in which both members of any given pair of rows have a one. The current state of this problem is the following: Finite projective planes are known to exist whenever m is the power of a prime and are known not to exist provided that $m \equiv 1$ or 2 mod 4 and that the square-free part of m has a prime factor congruent to 3 mod 4. In the particular case $m=6$ this had previously been shown by a direct attack, relying partly on numerical methods.* The question remains therefore unsettled for a sequence of values of m beginning with

$$m = 10, 12, 15, 20 \cdots .$$

The smallest of these is sufficient to present a formidable challenge, yet not so terrifying as to discourage an attack which combines mathematical ingenuity with the capabilities of high-speed automatic computing equipment.

The three basic methods of approach which have been tried are characteristically available in dealing with any problem of this kind, an ever-increasing number of which are presenting themselves in such connections as the theory of switching circuits and communication networks, the scheduling of production and transportation, as well as other assignment and allocation problems. Identifying them as the combinatorial, the algebraic, and the analytic approach, we shall illustrate each of them by summarizing how far it has carried towards solving our problem.

The combinatorial approach. The aim will be that of laying down principles or carrying through constructions which permit on the basis of succinct criteria

* G. Tarry, "Le Problème des 36 Officiers" Comptes Rendus, vol. 1, 1900, pp. 122–123 and vol. 2, 1901, pp. 170–203.

(or a relatively small number of trials) the elimination of large classes of candidates from the frequently astronomical number of possible ones. In the present instance, a matter of proper labelling makes it possible to reduce the problematical portion of A to a right lower corner which consists of an $m \times m$ array of mth order permutation matrices $c_{\rho\sigma}$ ($\rho, \sigma = 1, 2, \cdots, m$), subject to certain conditions resulting from the definition of a finite projective plane. It is possible, moreover, to arrange it so that this $m \times m$ array of matrices will be bordered on top and on the left by unit matrices. There now exists a suitable way of mapping mth order permutation matrices on column vectors. On assembling the column vectors thus corresponding to $c_{\rho 1}, c_{\rho 2}, \cdots, c_{\rho m}$ there results for each $\rho = 2, 3, \cdots, m$, a *latin square*. A set of $(m-1)$ latin squares arising in this manner from the incidence matrix of a finite projective plane consists of mutually orthogonal ones; that is, no ordered pair of elements occurs in identical positions more than once in any ordered pair of squares. *This reduces our problem to the existence problem for a set of* $(m-1)$ *mutually orthogonal latin squares of order* m, thus immersing it into the main stream of a large and important family of combinatorial problems which depend for their solution on the construction of permutation matrices with particular properties (*e.g.*, that of maximizing a suitable function) and which are encountered in the design of experiment as well as in assignment problems and their variants.

Tarry (*loc. cit.*) showed that for $m = 6$ there is not even a pair of such latin squares; Euler had already conjectured the same for any $m \equiv 2 \bmod 4$, but this is still an open question. Even the existence problem for a pair of mutually orthogonal ones of order 10 has resisted all efforts to date: programming the method of exhaustive search, which Tarry carried out by hand in the case $m = 6$, on a modern electronic computer shows it to be beyond the scope of current machines in spite of all improvements thus far devised; and the systematic construction of latin squares most prone to possess orthogonal companions has not as yet produced one which actually does.

The algebraic approach. An incidence matrix A can be characterized as a solution with non-negative integral elements of

$$AA^T = mI_n + U_n, \tag{1}$$

where I_n is the unit matrix and U_n has all its elements equal to 1. Relaxing the requirement that the elements of A be integral, one can find solutions of (1) with rational elements, satisfying at least some of the conditions which define an incidence matrix. For example, this definition can be satisfied as far as the first $(m+1)$ rows are concerned. As yet, however, there is little one can do to exploit these rational solutions in order to come nearer to the ultimate goal. Experiments were conducted, using the machine, in order to get information on the structure of a matrix G which would transform a rational solution S into an incidence matrix. If

$$T = SG$$

where S solves (1) and is normal, then T is also a normal solution of (1) provided G is orthogonal and its row and column sums are either all $+1$ or all -1. If T is integral it is an incidence matrix. Matrices G were therefore studied in cases where S and T are known, in the hope of finding ways of characterizing a manageably small class of possible G's applying also in unknown cases. So far this, too, has proved fruitless.

The analytic approach. This mode of attack consists in imbedding the discrete set of alternatives into a space of continuous variables. What is generally wanted is a procedure which, starting from an arbitrary point, lays down a "search path" on the now continuous manifold of candidates, leading to the desired object. For a computational approach there is required, in addition, a decision procedure when to break off such a search and from where to start anew, in case the original choice of a starting point has turned out to be a very bad one. In the case of our incidence matrix for the finite projective plane of order m, the conditions which define it characterize certain subspaces in the n^2-dimensional space of all matrices of order $n=m^2+m+1$; and the task becomes that of determining whether or not the intersection of these subspaces is empty. Hence the type of search procedures required is of the sort that, if it has failed to yield a common element at a certain stage, then it follows that there can be no such element. Computationally it has turned out more convenient to consider the existence of

$$Y = \frac{-1}{\sqrt{m}}\,(A - (m + \sqrt{m} + 1)^{-1}U_n)$$

in place of A, where Y is characterized as follows, if A is an incidence matrix:

(a) $$\sum_i Y_{ij} = \sum_j Y_{ij} = 1,$$

(b) $$-\frac{1}{\sqrt{m}(m+\sqrt{m}+1)} \leqq Y_{ij} \leqq \frac{\sqrt{m}+1}{m+\sqrt{m}+1},$$

(c) Y is orthogonal.

As yet it has not proved possible to determine whether or not these three subspaces have a non-empty intersection for $m=10$.

ON CHARACTERISTIC VALUE PROBLEMS IN HIGH ORDER DIFFERENTIAL EQUATIONS WHICH ARISE IN STUDIES ON HYDRODYNAMIC AND HYDROMAGNETIC STABILITY

S. CHANDRASEKHAR, University of Chicago

1. Introduction. Recent studies in hydrodynamic and hydromagnetic stability have disclosed the existence of a class of characteristic value problems in differential equations of high order—orders as high as twenty-four have been encountered—which appear to have a genuine mathematical interest. A partial list of these problems will be found in the Appendix. These problems have been solved, singly, as they have arisen in the physical connections. But one feels that there must be a general theory which embraces them all though such a theory is lacking at the present time. It is the object of this paper to bring these problems and the methods which have been developed for their solution to the attention of the mathematicians in the hope of stimulating their interest.

In the solution of the problems listed in the Appendix, methods of two different kinds have been found useful. In the first of these methods a variational procedure is developed which in the manner of its application is different from the usual ones. The second of these methods is based on expansion in orthogonal functions but again applied in an unusual manner. We shall illustrate the principles underlying these methods by considering one of the simpler problems listed in the Appendix.

2. A typical problem. G. I. Taylor's [1] investigation in 1923 on the stability of viscous flow between two concentric rotating cylinders provided the first example of a case of hydrodynamic instability for which the criterion was theoretically predicted and experimentally verified. The mathematical problem underlying this classic investigation in hydrodynamic stability is the following:

With certain simplifications suitable to the circumstances under which the experiments were performed, the problem requires the solution of the sixth order equation

$$(D^2 - a^2)^3 v = -a^2 T(1 + \alpha z)v, \tag{1}$$

with the boundary conditions

$$v = (D^2 - a^2)v = D(D^2 - a^2)v = 0 \tag{2}$$

for $z=0$ and 1, where T is the characteristic value parameter, $D=d/dz$ and a and α are assigned (real) constants. In the physical problem α is negative and one is interested in the range $0 \geqq \alpha \geqq -3.0$.

Equation (2) provides six boundary conditions (three at $z=0$ and three at $z=1$) and the requirement that a non-trivial solution of equation (1) satisfy these conditions will lead to a determinate sequence of possible values for T (for given a^2 and α). Among these possible values of T (for given α and varying a^2) there will be a smallest (positive) value; and in the physical problem particu-

lar interest is attached to the minimum of these smallest positive values of T as a function of a^2 (for fixed α).

In discussing the characteristic value problem presented by equations (1) and (2), we shall distinguish the cases $\alpha=0$ and $\alpha\neq 0$. The problem has basically different characters in the two cases: in the former case ($\alpha=0$) it is "self-adjoint" in some sense (yet to be defined!) while in the latter case it is not. Because of this difference the solution of the problem, while it can be made to depend on an extremal principle in the case $\alpha=0$, it cannot be so accomplished in the case $\alpha\neq 0$. As far as the general solution of equations (1) and (2) is concerned, a separate detailed discussion of the case $\alpha=0$ would hardly seem justified; but it does happen that the case $\alpha=0$ is the simplest proto-type of a wide class of problems to which similar methods of solution can be applied. For this reason we shall treat the case $\alpha=0$ at some length.

3. The case $\alpha=0$: example of the variational procedure. When $\alpha=0$, the solution of the characteristic value problem can, in principle, be achieved very simply. For, in this case, the solution of the equation

$$(D^2 - a^2)^3 v = -a^2 T v \tag{3}$$

must clearly be of the form

$$v_i = \sum_{i=1}^{6} A_i e^{q_i x}, \tag{4}$$

where the q_i's, occurring in pairs, are the roots of the characteristic equation

$$(q^2 - a^2)^3 = -a^2 T, \tag{5}$$

and the A_i's are constants of integration. The requirement that the solution represented by equation (4) satisfy the boundary conditions (2) will lead to a system of six linear homogeneous equations; and the determinant of this system must vanish if we are not to have the trivial solution $A_i=0$, $i=1, \cdots, 6$. And the condition that the determinant vanish will provide an equation for determining T. There is, of course, no difficulty of principle in carrying out this procedure. Indeed, it has been carried out by Pellew and Southwell [2] who find for example that

$$T = 1707.8 \qquad \text{for } a = 3.12. \tag{6}$$

(This is the value of a at which T, as a function of a, attains its minimum.)

For the case on hand the carrying out of the direct method of solution described in the preceding paragraph is particularly simple as the roots of equation (5) can be explicitly written down in terms of the known cube roots of -1; thus

$$q^2 = a^2 + \omega\sqrt[3]{(a^2 T)}, \qquad (\omega^3 = -1). \tag{7}$$

But this simplification does not arise in other problems which have to be considered; also, the characteristic equations which have to be solved are of higher order. It would therefore be useful if we can devise a method of solution which will avoid the necessity of solving complicated algebraic equations of high order by laborious methods of trial and error. This would be specially important if, as it often happens, we are required to establish the dependence of the characteristic values on two or more parameters rather than on a single parameter, such as a, in the present problem ($\alpha = 0$). And it does appear that in many of the cases which arise, convenient variational methods can be devised which we shall now illustrate by considering the problem presented by equations (2) and (3).

Letting

$$G = (D^2 - a^2)v \quad \text{and} \quad F = (D^2 - a^2)G = (D^2 - a^2)^2 v, \tag{8}$$

we can rewrite equation (3) in the form

$$(D^2 - a^2)F = -a^2 T v, \tag{9}$$

while the boundary conditions (2) require that

$$v = G = DG = 0 \qquad \text{for } z = 0 \text{ and } 1. \tag{10}$$

Let T_i denote a particular characteristic value and let the various functions derived from the solution v_i belonging to T_i be distinguished by a subscript i. Multiplying the equation governing v_i by G_j belonging to a different characteristic value T_j and integrating the resulting equation over the range of z, we obtain

$$\int_0^1 G_j(D^2 - a^2)F_i dz = -a^2 T_i \int_0^1 v_i(D^2 - a^2)v_j dz. \tag{11}$$

The integral occurring on the left-hand side of this equation can be reduced by two successive integration by parts; thus,

$$\int_0^1 G_j(D^2 - a^2)F_i dz = [G_j DF_i - (DG_j)F_i]_0^1 + \int_0^1 F_i(D^2 - a^2)G_j dz. \tag{12}$$

The integrated parts vanish in virtue of the boundary conditions (*cf.* equations (10)) and we are left with

$$\int_0^1 G_j(D^2 - a^2)F_i dz = \int_0^1 F_i F_j dz. \tag{13}$$

Similarly, after an integration by parts the right-hand side of equation (11) becomes

$$a^2 T_i \int_0^1 [(Dv_i)(Dv_j) + a^2 v_i v_j] dz. \tag{14}$$

Thus,

$$\int_0^1 F_i F_j dz = a^2 T_i \int_0^1 [(Dv_i)(Dv_j) + a^2 v_i v_j] dz. \tag{15}$$

Noticing that interchanging i and j in this equation replaces T_i by T_j but leaves the equation otherwise unaffected, we conclude that

$$\int_0^1 F_i F_j dz = 0 \qquad i \neq j; \tag{16}$$

and, further, that when $i=j$ the corresponding characteristic value, T, can be expressed as the ratio of two positive definite integrals in the form:

$$T = \frac{\int_0^1 F^2 dz}{a^2 \int_0^1 [(Dv)^2 + a^2 v^2] dz}. \tag{17}$$

These facts clearly imply the existence of an extremal principle which can be made the basis of a variational method of solving the underlying characteristic value problem. For comparison with the method we shall describe in §4 for the case $\alpha \neq 0$, we shall formulate the extremal principle in the context of a slightly transformed equation.

Operating on equation (3) by (D^2-a^2) we get

$$(D^2 - a^2)^2 W = -a^2 T W \tag{18}$$

as the equation governing

$$W = (D^2 - a^2)v. \tag{19}$$

According to equations (2) and (3) the corresponding boundary conditions on W are:

$$W = DW = 0 \quad F \equiv (D^2 - a^2)G \equiv (D^2 - a^2)^2 W = 0 \qquad \text{for } z = 0 \text{ and } 1. \tag{20}$$

Again rewriting equation (18) in the form

$$(D^2 - a^2)F = -a^2 T W, \tag{21}$$

and multiplying the equation governing W_i (belonging to T_i) by F_j (belonging to T_j) we obtain

$$\int_0^1 F_j (D^2 - a^2) F_i dz = -a^2 T_i \int_0^1 W_i (D^2 - a^2) G_j dz. \tag{22}$$

Making use of the boundary conditions (20) we can reduce equation (22) by one or more integrations by parts to the form

$$\int_0^1 [(DF_i)(DF_j) + a^2F_iF_j]dz = a^2T_i\int_0^1 G_iG_jdz. \tag{23}$$

From this equation it follows that

$$\int_0^1 G_iG_jdz = 0, \qquad i \neq j; \tag{24}$$

and that when $i=j$ we can express T as:

$$T = \frac{\int_0^1 [(DF)^2 + a^2F^2]dz}{a^2\int_0^1 G^2dz}. \tag{25}$$

This last formula, like (17), expresses T as the ratio of two positive definite integrals.

Consider now the effect on T (evaluated according to equation (25)) of an arbitrary variation, δW, in W compatible only with the boundary conditions on W. We find in a straightforward manner that

$$\delta T = -\frac{2}{a^2\int_0^1 G^2dz}\int_0^1 \delta F\{(D^2 - a^2)F + a^2TW\}dz, \tag{26}$$

where it should be noted that in the reductions leading to equation (26) the relations implied in the definitions of F and G (*cf.* equation (20)) have been used; in particular

$$\delta F = (D^2 - a^2)^2\delta W. \tag{27}$$

Hence, to the first order, $\delta T \equiv 0$ for all small arbitrary variations in W which satisfy the boundary conditions, provided

$$(D^2 - a^2)F = -a^2TW, \tag{28}$$

i.e., if the differential equation governing W is satisfied. It is evident that the converse of this proposition is also true. Further, it follows from (25) that the true solution of the problem (belonging to the lowest characteristic value T) leads to the minimum value (in the sense of the calculus of variations) for T when evaluated according to (25).

While the minimal principle formulated as above is valid for all small arbitrary variations δW compatible only with the boundary conditions on W, it is true also for all small arbitrary variations δF, the variation, δW, in W being determined in terms of δF by the equation

$$(D^2 - a^2)^2\delta W = \delta F, \tag{29}$$

and the boundary conditions

$$\delta W = D\delta W = 0 \qquad \text{for } z = 0 \text{ and } 1. \tag{30}$$

The validity of this minimal principle in this more restricted form actually provides a more effective basis for a variational procedure for determining T. For on this latter basis the procedure would be the following:

Assume for F an expression involving one or more parameters, A_k, and which vanishes at $z=0$ and 1. With the chosen form of F solve the equation

$$(D^2 - a^2)^2 W = F \tag{31}$$

for W and arrange that the solution satisfies the boundary conditions $W=DW=0$ for $z=0$ and 1; since equation (31) is of the fourth order there will be just enough constants of integration to do this. With W determined in this fashion evaluate T according to (25) and minimize it with respect to the parameters A_k. In this way we shall obtain the "best" value of T for the chosen form of F.

In practice it is found that even with the simplest trial functions, the variational procedure in the foregoing form gives surprisingly high accuracy in the deduced values of T. Thus, with the trial function, $F=\sin \pi z$, with no variational parameter, the method gives $T=1715.1$ for $a=3.12$; while the function, $F=\sin \pi z+A \sin 3\pi z$, with one variational parameter leads to $T=1707.9$ for the same value of a; these values obtained in the "first" and the "second" approximations should be compared with the value $T=1707.8$ obtained from an "exact" solution of the problem. The origin of this high precision in the deduced values of T must clearly be traced to the fact that in satisfying four of the six boundary conditions of the problem (namely $W=DW=0$ for $z=0$ and 1) we have *exactly* satisfied the second of the pair of differential equations,

$$(D^2 - a^2)F = -a^2 TW \quad \text{and} \quad (D^2 - a^2)^2 W = F, \tag{32}$$

which governs the problem.

[It may be noticed here that had we used equation (17) (instead of (25)) as the basis of the variational method, we should have had to assume a form for DG such that not only does it vanish at $z=0$ and 1, but also G, obtained after integration, vanishes at the same points; further, W will have to be obtained as the solution of $(D^2-a^2)W=G$ which vanishes at $z=0$ and 1. Apart from these differences in detail the application of the variational method based on (17) proceeds along essentially the same lines. But the method based on (25) is preferable since it avoids the restrictions implied by the requirement that both G and DG vanish at $z=0$ and 1.]

4. The case $\alpha \neq 0$: example of a method based on expansion in orthogonal functions when no extremal principle exists. Returning to equation (1) and the general case $\alpha \neq 0$, we can readily verify that propositions similar to those embodied in equations (15), (16) and (17), for the case $\alpha=0$, cannot be established now. Thus, if equation (1) and the boundary conditions (2) together represent

a system which might be described as "self-adjoint" when $\alpha=0$, it cannot be so described when $\alpha\neq 0$. It is important to note that this conclusion in no way depends on the sign of α; in particular it is independent of the circumstance that when $\alpha<-1$ the operator whose characteristic values we are seeking becomes singular on account of $(1+\alpha z)$ having a zero in the range $0\leqq z\leqq 1$.

Since equations (1) and (2), in general, do not allow the formulation of an extremal principle on which a variational procedure might be based and since the general solution of equation (1) cannot also be written down in any convenient form, it would appear that the only remaining course is to expand the unknown functions in Fourier series. In Taylor's original investigation this was the method which was adopted: Taylor expanded v in a sine series of the form

$$v=\sum_{n=1}^{\infty} V_n \sin n\pi z \tag{33}$$

and obtained T as the characteristic root of an infinite matrix. But the process of determining the root was not a very convergent one; and he succeeded (with considerable effort) in determining the characteristic root for only one value of $\alpha<-1$. However, we may expect that a more rapidly convergent process will be obtained if in using the method of expansion in Fourier series, we incorporate in the method the same basic idea which led to the high precision of the variational method for the case $\alpha=0$. We shall now indicate how this can be accomplished (see Chandrasekhar [3]).

First we transform equation (1) by rewriting it in the form

$$\frac{1}{1+\alpha z}(D^2-a^2)^2 v=-a^2 T v, \tag{34}$$

and operating on it by (D^2-a^2); in this way, we obtain the differential equation

$$(D^2-a^2)\left\{\frac{1}{1+\alpha z}(D^2-a^2)^2 W\right\}=-a^2 T W, \tag{35}$$

for

$$W=(D^2-a^2)v. \tag{36}$$

The corresponding boundary conditions on W are the same as those given in (20).

Equation (35) is equivalent to the pair of equations

$$(D^2-a^2)^2 W=(1+\alpha z)\psi \tag{37}$$

and

$$(D^2-a^2)\psi=-a^2 T W, \tag{38}$$

while the boundary conditions (20) can be expressed alternatively in the forms

$$W = DW = 0 \quad \text{and} \quad \psi = 0 \qquad \text{for } z = 0 \text{ and } 1. \tag{39}$$

Since ψ has to vanish at $z=0$ and 1, we can expand it in a sine series of the form

$$\psi = \sum_{n=1}^{\infty} C_n \sin n\pi z. \tag{40}$$

Having chosen ψ in this manner, we next *solve* the equation

$$(D^2 - a^2)^2 W = (1 + \alpha z) \sum_{n=1}^{\infty} C_n \sin n\pi z, \tag{41}$$

obtained by inserting (40) in (37), and arrange that the solution satisfies the four remaining conditions on W; since equation (41) is of the fourth order there will be just enough constants of integration to do this. With W determined in this fashion and ψ given by (40), equation (38) will lead to an infinite determinant which must be zero if all the C_n's are not to vanish. In this way we shall obtain a characteristic equation for determining T.

When the details of the method described in the preceding paragraphs are carried out, one finds that the process of solving the infinite order characteristic equation for T, by setting the determinant formed by the first n rows and columns equal to zero and letting n take increasingly larger values, converges very rapidly indeed. Thus for $\alpha = -2.5$ and $a = 5.00$, 5.05 and 5.10 the values of T obtained in the third and the fourth approximations (the "order" of the approximation being the order of the determinant which is set equal to zero in the determination of T) are:

$$a = \begin{cases} 5.00 \\ 5.05 \\ 5.10 \end{cases} \quad T \text{ (3rd app.)} = \begin{cases} 4.607 \times 10^4 \\ 4.600 \times 10^4 \\ 4.604 \times 10^4 \end{cases} \quad T \text{ (4th app.)} = \begin{cases} 4.626 \times 10^4 \\ 4.619 \times 10^4. \\ 4.623 \times 10^4 \end{cases} \tag{42}$$

It is seen that the values of T given in the third and the fourth approximations differ by only four parts in a thousand. The origin of this rapid convergence clearly lies in the splitting of the original equation of order six into a pair of order two and four respectively and satisfying the equation of order four exactly. This basic idea underlying the method is capable of extension and application to a wide class of problems.

Appendix

We shall list here some of the more important characteristic value problems in high order differential equations which have occurred in recent studies on hydrodynamic and hydromagnetic stability. The particular physical connections in which they arise are indicated. The references are to papers in which the solutions of the problems will be found.

I. *The inhibition of convection by a magnetic field* [4, 5].

1) To solve

$$(D^2 - a^2)[(D^2 - a^2)^2 - QD^2]W = -a^2RW,$$

together with the boundary conditions

$$W = [(D^2 - a^2)^2 - QD^2]W = 0 \quad \text{on} \quad z = \pm \tfrac{1}{2},$$

and

$$\begin{aligned} &\textit{either} && DW = 0 \quad \text{on} \quad z = \pm \tfrac{1}{2}, \\ &\textit{or} && D^2W = 0 \quad \text{on} \quad z = \pm \tfrac{1}{2}, \\ &\textit{or} && DW = 0 \quad \text{on} \quad z = +\tfrac{1}{2} \quad \text{and} \quad D^2W = 0 \quad \text{on} \quad z = -\tfrac{1}{2}, \end{aligned}$$

where a and Q are assigned positive constants and R is the characteristic value parameter.

The physical problem requires the minimum (with respect to a) of the lowest characteristic value R for various assigned values of Q.

2) To solve

$$(D^2 - a^2)[(D^2 - a^2)^2 - Q(D + i\tilde{\omega})^2]W = -a^2RW,$$

together with the boundary conditions

$$|W| = |(D^2 - a^2)^2W - Q(D + i\tilde{\omega})^2W| = 0 \quad \text{on} \quad z = \pm \tfrac{1}{2},$$

and

$$\begin{aligned} &\textit{either} && |DW| = 0 \quad \text{on} \quad z = \pm \tfrac{1}{2}, \\ &\textit{or} && |D^2W| = 0 \quad \text{on} \quad z = \pm \tfrac{1}{2}, \\ &\textit{or} && |DW| = 0 \quad \text{on} \quad z = +\tfrac{1}{2} \quad \text{and} \quad |D^2W| = 0 \quad \text{on} \quad z = -\tfrac{1}{2}, \end{aligned}$$

where a, $\tilde{\omega}$ and Q are assigned constants and R is the characteristic value parameter.

The physical problem requires the minimum (with respect to a and $\tilde{\omega}$) of the lowest characteristic value R for various assigned values of Q. Since W is complex, we have here a genuine problem in an equation of order twelve.

II. *The instability of a layer of fluid heated below and subject to Coriolis acceleration* [6].

3) To solve

$$(D^2 - a^2)Z = -\left(\frac{2\Omega}{\nu}\, d\right)DW$$

and

$$(D^2 - a^2)\left[(D^2 - a^2)^2W - \left(\frac{2\Omega}{\nu}\, d^3\right)DZ\right] = [(D^2 - a^2)^3 + TD^2]W = -a^2RW,$$

together with the boundary conditions

$$W = (D^2 - a^2)^2 W - \left(\frac{2\Omega}{\nu} d^3\right) DZ = 0 \quad \text{on} \quad z = \pm \tfrac{1}{2},$$

and

either $DW = Z = 0$ on $z = \pm \frac{1}{2}$,

or $D^2W = DZ = 0$ on $z = \pm \frac{1}{2}$,

or $DW = Z = 0$ on $z = +\frac{1}{2}$ and $D^2W = DZ = 0$ on $z = -\frac{1}{2}$,

where a, ν, d and Ω are assigned constants, $T = 4\Omega^2 d^4/\nu^2$ and R is the characteristic value parameter.

The physical problem requires the minimum (with respect to a) of the lowest characteristic value R for various assigned values of T.

3a) To solve

$$(D^2 - a^2 - i\sigma)Z = -\left(\frac{2\Omega}{\nu} d\right) DW,$$

$$(D^2 - a^2)(D^2 - a^2 - i\sigma)W - \left(\frac{2\Omega}{\nu} d^3\right) DZ = F$$

and

$$(D^2 - a^2 - i\tilde{\omega}\sigma)F = -Ra^2 W,$$

together with the boundary conditions

$$|F| = |W| = 0 \quad \text{on} \quad z = \pm \tfrac{1}{2},$$

and

either $DW = Z = 0$ on $z = \pm \frac{1}{2}$,

or $D^2W = DZ = 0$ on $z = \pm \frac{1}{2}$,

or $DW = Z = 0$ on $z = +\frac{1}{2}$ and $D^2W = DZ = 0$ on $z = -\frac{1}{2}$,

where $\tilde{\omega}$ is a further assigned constant and σ (which is real) is a parameter to be determined (for given $T = 4\Omega^2 d^4/\nu^2$ and a^2) by the condition that R is real; and the physical problem requires the minimum (with respect to a^2) of these real values of R for various assigned values of T and $\tilde{\omega}$.

III. *The instability of a layer of fluid heated below and subject simultaneously to a magnetic field and Coriolis acceleration* [7].

4) To solve

$$(D^2 - a^2)[\{(D^2 - a^2)^2 - QD^2\}^2 + TD^2(D^2 - a^2)]W = -Ra^2[(D^2 - a^2)^2 - QD^2]W$$

together with (several different) five pairs of boundary conditions on W at

$z = \pm \frac{1}{2}$, where a, Q and T are assigned constants and R is the characteristic value parameter.

IV. *The stability of viscous flow between rotating cylinders* [1, 3, 8, 9].

5) To solve

$$(D^2 - a^2)^2 v = -a^2 T(1 + \alpha z)v,$$

together with the boundary conditions

$$v = D^2 v = D(D^2 - a^2)v = 0 \quad \text{on} \quad z = 0 \quad \text{and} \quad 1,$$

where a and α (<0) are assigned constants and T is the characteristic value parameter. The physical problem requires the minimum (with respect to a) of the lowest (positive) characteristic value T for various assigned values of $\alpha(<0)$.

6) To solve

$$(DD^* - \lambda^2)^2 v = \frac{4A\lambda^2}{\nu^2}\left(A + \frac{B}{r^2}\right)v,$$

where $D = d/dr$ and $D^* = D + 1/r$, together with the boundary conditions

$$v = (DD^* - \lambda^2)v = D^*(DD^* - \lambda^2)v = 0 \qquad \text{for } r = R_1 \text{ and } R_2;$$

λ, ν and B are assigned constants and A is the characteristic value parameter.

V. *The stability of viscous flow between rotating cylinders in the presence of a magnetic field* [10].

7) To solve

$$[(D^2 - a^2)^2 + Qa^2]^2 \psi = -Ta^2(D^2 - a^2)\psi,$$

together with the boundary conditions

$$D\psi = (D^2 - a^2)\psi = [(D^2 - a^2)^2 + Qa^2]\psi = D[(D^2 - a^2)^2 + Qa^2]\psi = 0 \text{ on } z = \pm \tfrac{1}{2},$$

where a and Q are assigned constants and T is the characteristic value parameter. The physical problem requires the minimum (with respect to a) of the lowest characteristic value T for various assigned values of Q.

VI. *The stability of viscous flow between rotating cylinders in the presence of a radial temperature gradient* [11].

8) To solve

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right)^2 W = -n^2 S_n W,$$

together with the boundary conditions

$$W = \frac{dW}{dr} = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right)^2 W = 0 \qquad \text{for } r = 1 \text{ and } \eta(<1),$$

where n is an integer and S_n is the characteristic value parameter.

9) To solve

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r}\right)^3 W = -n^2 S_n \frac{W}{r^2},$$

together with the same boundary conditions as in (8) above.

VII. *The thermal instability of fluid spheres and spherical shells* [12, 13, 14].

10) To solve

$$\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}\right)^3 W = -l(l+1)C_l W,$$

together with the boundary conditions

(a) $$W = \left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}\right)^2 W = 0$$

and

$$\textit{either} \quad \frac{dW}{dr} \quad \textit{or} \quad \frac{d^2W}{dr^2} = 0 \qquad \text{for } r = 1,$$

and

$$W = O(r^l) \qquad \text{as } r \to 0;$$

(b) $$W = \frac{d^2W}{dr^2} = \left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}\right]^2 W = 0 \qquad \text{for } r = 1 \text{ and } \eta(<1),$$

where l is an integer and C_l is the characteristic value parameter.

VIII. *The stability of superposed fluids (plane problem)* [15, 16, 17, 18, 19].

11) To solve

$$D\left\{\left[\rho+\frac{\mu}{\sigma}(-k^2+D^2)\right]Dw+\frac{1}{\sigma}(D\mu)(k^2+D^2)w\right\}$$
$$= k^2\left\{\left[-\frac{g}{\sigma^2}D\rho+\rho+\frac{\mu}{\sigma}(-k^2+D^2)\right]w+\frac{2}{\sigma}(D\mu)(Dw)\right\},$$

together with the boundary conditions

$$w = 0 \quad \text{for} \quad z = 0 \quad \text{and} \quad l$$

and

either	$Dw = 0$	for	$z = 0$	and	l,			
or	$D^2w = 0$	for	$z = 0$	and	l,			
or	$Dw = 0$	for	$z = 0$	and	$D^2w = 0$	for	$z = l$,	
or	$D^2w = 0$	for	$z = 0$	and	$Dw = 0$	for	$z = l$,	

where $\rho=\rho(z)$ and $\mu=\mu(z)$ are given functions of z, k is an assigned (real) constant and σ is the characteristic value parameter. (Note that σ can be complex).

IX. *The stability of superposed fluids (spherical problem)* [20].

12) To solve

$$\frac{d}{dr}\left\{\rho\frac{d}{dr}(rW)+\frac{\mu}{\sigma}\frac{d}{dr}(rF)\right\}+\frac{1}{\sigma}\frac{d}{dr}\left\{r\frac{d\mu}{dr}\left[\frac{d^2W}{dr^2}+\frac{(l+2)(l-1)}{r^2}W\right]\right\}$$
$$-\frac{l(l+1)}{r}\left(\rho W+\frac{\mu}{\sigma}F\right)=-\frac{l(l+1)}{\sigma^2}\left\{\gamma\frac{d\rho}{dr}W-2\sigma\frac{d\mu}{dr}\frac{d}{dr}\left(\frac{W}{r}\right)\right\},$$

where

$$F=\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}\right]W,$$

together with the boundary conditions

$$W=0 \quad \text{and} \quad \textit{either } \frac{dW}{dr} \textit{ or } \frac{d^2W}{dr^2}=0 \quad \text{on} \quad r=R,$$

and

$$W=O(r^l) \qquad \text{as } r\to 0.$$

In the foregoing ρ, μ and γ are given functions of r, l is an integer and σ (which may be complex) is the characteristic value parameter.

The case when

$$\mu=\mu_1=\text{constant}, \qquad \rho=\rho_1=\text{constant for } 0\leqq r<r_i$$

and

$$\mu=\mu_2=\text{constant}, \qquad \rho=\rho_2=\text{constant for } r_i<r\leqq R$$

is of particular importance. In this case the equation reduces to the form

$$\frac{d^2}{dr^2}\left\{r\left(W+\frac{\mu}{\rho\sigma}F\right)\right\}-\frac{l(l+1)}{r}\left(W+\frac{\mu}{\rho\sigma}F\right)=0$$

in each of the two regions of constant ρ and μ. And solutions are sought which satisfy the boundary conditions

$$W=0 \quad \text{and} \quad \frac{d^2W}{dr^2}=0 \qquad \text{for } r=R$$

$$W=O(r^l) \qquad \text{for } r\to 0$$

and

$$W, \quad \frac{dW}{dr} \quad \text{and} \quad \mu\left\{r^2\frac{d^2W}{dr^2} + (l+2)(l-1)W\right\}$$

are continuous on $r = r_i$.

References

1. G. I. Taylor, Stability of a viscous liquid contained between two rotating cylinders, Philos. Trans. Roy. Soc. London Ser., A, vol. 223, 1923, pp. 289–343.
2. A. Pellew and R. V. Southwell, On maintained convective motion in a fluid heated from below, Proc. Roy. Soc. London Ser. A, vol. 176, 1940, pp. 312–343.
3. S. Chandrasekhar, The stability of viscous flow between rotating cylinders, Mathematika (in press).
4. S. Chandrasekhar, On the inhibition of convection by a magnetic field, Philos. Mag. Ser. 7, vol. 43, 1952, pp. 501–532.
5. S. Chandrasekhar, On the inhibition of convection by a magnetic field. II, Philos. Mag. (in press).
6. S. Chandrasekhar, The instability of a layer of fluid heated below and subject to Coriolis forces, Proc. Roy. Soc. London Ser. A, vol. 217, 1953, pp. 306–327; also a further paper in press.
7. S. Chandrasekhar, The instability of a layer of fluid heated below and subject to the simultaneous action of a magnetic field and rotation, Proc. Roy. Soc. London Ser. A, (in press).
8. D. Meksyn, Stability of viscous flow between rotating cylinders. I, II, III, Proc. Roy. Soc. London Ser. A, vol. 187, 1946, pp. 115–128, 480–491, 492–504.
9. S. Chandrasekhar, (in press).
10. S. Chandrasekhar, The stability of viscous flow between rotating cylinders in the presence of a magnetic field, Proc. Roy. Soc. London Ser. A, vol. 216, 1953, pp. 293–309.
11. S. Chandrasekhar, The stability of viscous flow between rotating cylinders in the presence of a radial temperature gradient, Journal of Rational Mechanics and Analysis, vol. 3, 1954, pp. 181–207.
12. H. Jeffreys and M. E. M. Bland, The instability of a fluid sphere heated within, Mon. Not. R. Astr. Soc. Geophys. Sup. vol. 6, 1951, pp. 148–158.
13. S. Chandrasekhar, The thermal instability of a fluid sphere heated within, Philos. Mag. Ser. 7, vol. 43, 1952, pp. 1317–1329.
14. S. Chandrasekhar, The onset of convection by thermal instability in spherical shells, Philos. Mag. Ser. 7, vol. 44, 1953, pp. 233–241, 1129–1130.
15. Lord Rayleigh, Investigation of the character of the equilibrium of an incompressible heavy fluid of variable density, Scientific Papers, vol. 2, Cambridge University Press, Cambridge, 1900, pp. 200–207.
16. G. I. Taylor, The instability of liquid surfaces when accelerated in a direction perpendicular to their planes. I, Proc. Roy. Soc. London Ser. A, vol. 201, 1950, pp. 192–196; also D. J. Lewis, The instability of liquid surfaces when accelerated in a direction perpendicular to their planes. II, Proc. Roy. Soc. London Ser. A, vol. 202, 1950, pp. 81–96.
17. S. Chandrasekhar, (in press).
18. R. Hide, (in press).
19. Tsung-Ying Teng Fan, (in press).
20. S. Chandrasekhar, The character of the equilibrium of an incompressible fluid sphere of variable density and viscosity subject to radial acceleration, Quart. Jour. of Mechanics and Applied Mathematics, (in press).

FREQUENCY SPECTRUM OF VIBRATIONS OF A CRYSTAL LATTICE

ELLIOTT W. MONTROLL, University of Maryland,* and Office of Naval Research

1. Introduction. The heat capacity of a system of N interacting particles under an external constraint x is defined as $C_x = (\partial Q/\partial T)_x$, the ratio of the added heat to the temperature rise of the system. Experimentally it is easiest to measure the heat capacity of solids at constant pressure, C_p, and of gases at constant volume (in a closed container), C_v. The quantities C_p and C_v are related through the thermodynamic equation $C_p - C_v = 9\alpha^2 VT/\kappa$, α being the coefficient of thermal expansion, V the volume, and κ the compressibility. Since statistical mechanical formulae lead naturally to C_v, one must include the correction due to lattice expansion in applying theoretical results to the interpretation of experimental C_p measurements in solids.

The modern theory of the heat capacity of solids was originated by Einstein [1] in 1907 through the application of the then new Planck quantum theory. The appropriate basic formulae are those for the internal energy and heat capacity of N independent harmonic oscillators of frequency ν:

$$E = 3Nh\nu\left[\frac{1}{2} - \frac{1}{1 - \exp(h\nu/kT)}\right] \tag{1.1}$$

$$C_v = 3Nk\left(\frac{1}{2}\frac{h\nu}{kT}\right)^2 \Big/ \sinh^2\left(\frac{1}{2}\frac{h\nu}{kT}\right). \tag{1.2}$$

Einstein assumed that each atom in a solid vibrates about its equilibrium position independently of its neighbors and used equation (1.2) to fit the then available heat capacity data. Equation (1.2) is a universal function of the variable $kT/h\nu$. Hence it was predicted that all experimental C_v curves, when plotted on an appropriate scale, could be superimposed. The frequency ν determined the scale. The anomolously low value of the atomic heat of diamond was explained by stating that its frequency ν was larger than that of most materials.

The general form of experimental C_v curves is given in Figure 1. Experimental deviations from the Einstein formula are to be expected at low temperatures (and indeed have been observed for years) because atoms in a solid do not vibrate independently of each other. Interatomic forces are such that the oscillations of a given atom are coupled with those of its neighbors. Hence, the $3N$ equal frequencies of equation (1.2) should be replaced by the set of frequencies of the normal modes of the coupled oscillating atoms.

* The preparation of this article was partially supported by the Office of Scientific Research of the U. S. Air Force.

It is useful to introduce a distribution function $g(\nu)$ which has the property that $g(\nu)d\nu$ is the number of normal modes between ν and $\nu+d\nu$. Then equations (1.1) and (1.2) become

$$E = \int_0^{\nu_L} h\nu \left[\frac{1}{2} - \frac{1}{1 - \exp(h\nu/kT)}\right] g(\nu)d\nu \tag{1.3}$$

and

$$C_v = k\int_0^{\nu_L} g(\nu)\left[\left(\frac{1}{2}\frac{h\nu}{kT}\right)^2 \Big/ \sinh^2\left(\frac{1}{2}\frac{h\nu}{kT}\right)\right] d\nu. \tag{1.4}$$

Here ν_L is the largest normal mode frequency. Optical as well as thermodynamic properties of crystals can be discussed in terms of $g(\nu)$.

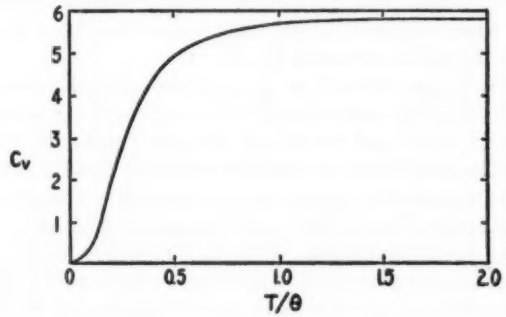


FIG. 1. The variation of the specific heat (in calories per gram mol) of a typical solid as a function of temperature.

The first calculation of the distribution of frequencies of normal modes was made by Debye in 1912. He postulated a solid to be an elastic continuum. Since a continuum has an infinite number of normal modes, Debye cut the frequency spectrum off at a frequency ν_L such that the total number of normal modes was equal to the number of degrees of freedom of the solid. In the Debye [2] theory $g(\nu) = 9N(\nu/\nu_L)^2/\nu_L$ where ν_L, the largest frequency, is a function of the elastic constants of a solid; $g(\nu) = 0$ when $\nu > \nu_L$. Debye's work was one of the great successes of the early quantum theory. The theoretical heat capacities based on his frequency spectrum are in good agreement with experimental results. The simplicity of the Debye theory combined with this fact has given it a long life as the dominant theory of the heat capacity of solids.

The continuum model can be expected to give good results at very low temperatures where only long wave length oscillations are important. However, in the steep part of the C_v curve it is not sufficiently accurate. A discussion of the experimental situation is given in a review by Blackman [3]. Born and von Karman [4] were the first to investigate the dynamics of periodically spaced

atoms. They introduced a model of a solid which is mechanically equivalent to a set of coupled springs and masses.

Our main purpose here is to review the general characteristics of the frequency distribution, $g(\nu)$, of the Born-Karman model. A striking feature of $g(\nu)$ curves is the existence of certain singularities, logarithmic ones in two dimensional lattices and inverse square root singularities in the derivative of $g(\nu)$ in three-dimensional lattices. The existence of peaks was first noticed by Blackman. The analytical nature of these peaks, our main subject of interest here, was first discussed in an investigation [5] of the vibrations in the plane of a two dimensional square lattice with interactions between atoms which were nearest and next nearest neighbors. The logarithmic peaks which occurred in the $g(\nu)$ of this case were also found by Rosenstock and Bowers [6] for the transverse vibrations of atoms in a square lattice and finally by Smollett [7] for the in plane vibrations of the two dimensional lattices with long range Coulomb interactions. Similar results were obtained for the transverse vibrations in the two dimensional hexagonal graphite structure [8, 9].

Van Hove [10] then showed by an application of the critical point theory of M. Morse [11] that the existence of these singularities is a consequence of the periodic lattice structure and not of the detailed character of the atomic force constants. He discussed both two and three dimensional lattices. His theory of the nature of the singularities in $g(\nu)$ for the three dimensional case was verified in specific examples by Newell [12] and Rosenstock [13, 14].

We shall use the abbreviations 2D and 3D in their usual connotation and FS to mean frequency spectrum. RL will denote reciprocal lattice.

Many of the results to be reviewed here are applicable to the general problem of wave propagation in 2D and 3D periodic structures. For example the electron energy level distribution in metals has the same behavior as the $g(\nu)$ of lattice vibrations.

2. Fundamental formulae for analysis of doubly periodic lattices. We shall introduce the discussion of the vibration of periodic lattices by analyzing some 2D examples and then generalizing to 3D systems.

Let us consider a square lattice which contains N rows and N columns of identical particles of mass M. The lattice points are identified by (l,m) (where both l and m range through the integers $1, 2, \cdots, N$). The components of the displacement of the particle (l,m) are designated by $u_{l,m}$ and $v_{l,m}$. Our notation is summarized in Figure 2.

We shall assume that our lattice is formed on a torus so that we can apply the simplifying Born-Karman boundary conditions:

$$u_{l,m} = u_{l+N,m}, \qquad v_{l,m} = v_{l+N,m} \tag{2.1a}$$

$$u_{l,m} = u_{l,m+N}, \qquad v_{l,m} = v_{l,m+N}. \tag{2.1b}$$

Physically one expects thermodynamic functions which are proportional to the number of particles in a system to be independent of surface effects in three

dimensions and of boundary effects in two dimensions. We shall discuss the mathematical theory of this point in section 5. The boundary conditions (2.1) are generally used in solid state problems.

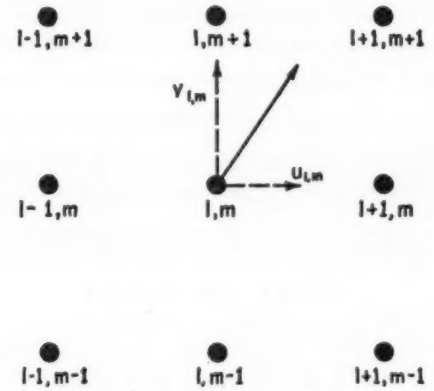


FIG. 2. Designation of lattice points in a two dimensional lattice.

When the particles are displaced from their equilibrium positions the total potential energy of interactions can be expanded in a Taylor series about its equilibrium value:

$$\Phi = \Phi_0 - \sum C_{l,m}u_{l,m} - \sum C'_{l,m}v_{l,m} + \text{terms quadratic, cubic, } \textit{etc.}, \text{ in } u_{l,m} \text{ and } v_{l,m}.$$

Since $C_{l,m}$ and $C'_{l,m}$ represent the forces in the horizontal and vertical directions acting on particle (l, m) while it is at equilibrium, they must vanish, equilibrium being defined as that state in which no forces act on any particles. By making the hypothesis that displacements from equilibrium are small, we are justified in neglecting the cubic and higher order terms in the displacements.

The forces which act in the horizontal and vertical directions respectively on the (l,m)-th particle are

$$-\frac{\partial \Phi}{\partial u_{l,m}} = -\sum_{\lambda,\mu} a_{\lambda,\mu}u_{l+\lambda,m+\mu} - \sum_{\lambda,\mu} b_{\lambda,\mu}v_{l+\lambda,m+\mu} \tag{2.2a}$$

$$-\frac{\partial \Phi}{\partial v_{l,m}} = -\sum_{\lambda,\mu} a'_{\lambda,\mu}u_{l+\lambda,m+\mu} - \sum_{\lambda,\mu} b'_{\lambda,\mu}v_{l+\lambda,m+\mu} \tag{2.2b}$$

where $-a_{\lambda,\mu}u_{l+\lambda,m+\mu}$ is the contribution of the horizontal force on (l,m) due to a horizontal displacement of a particle λ lattice distances to the right and μ above (l,m), *etc.*

The equations of motion of our system of coupled particles are

$$M d^2u_{l,m}/dt^2 = -\partial\Phi/\partial u_{l,m} \tag{2.3a}$$

$$M d^2v_{l,m}/dt^2 = -\partial\Phi/\partial v_{l,m}. \tag{2.3b}$$

These equations are linear in the displacements and contain only constant coefficients to the $u_{l,m}$'s and $v_{l,m}$'s. Hence we can seek solutions of the form

$$u_{l,m} = ue^{2\pi i\nu t}e^{i(\phi_1 l+\phi_2 m)} \tag{2.4a}$$

$$v_{l,m} = ve^{2\pi i\nu t}e^{i(\phi_1 l+\phi_2 m)} \tag{2.4b}$$

where u, v, ν, ϕ_1, and ϕ_2 are constants which must be chosen so that (2.4) satisfies (2.3), the boundary conditions (2.1), and whatever initial conditions that might be imposed.

The boundary conditions (2.1) are satisfied if ϕ_1 and ϕ_2 are solutions of

$$e^{i\phi_1 N} = 1 \quad \text{and} \quad e^{i\phi_2 N} = 1,$$

that is, if

$$\phi_1 = 2\pi k/N \quad \text{and} \quad \phi_2 = 2\pi j/N \tag{2.5}$$

with k, $j = 1, 2, \cdots, N$ or (if N is even) k, j may run through $-\frac{1}{2}N+1$, $-\frac{1}{2}N+2, \cdots, 0, 1, 2, \cdots, \frac{1}{2}N$. This gives N^2 possible choices of the pair (ϕ_1, ϕ_2). The frequency ν is related to ϕ_1 and ϕ_2 by noting that if (2.4) is substituted into (2.2) and (2.3) the constants u and v are solutions of the homogeneous linear equations

$$u\{F_{11}(\phi_1, \phi_2) - 4\pi^2\nu^2M\} + vF_{12}(\phi_1, \phi_2) = 0 \tag{2.6a}$$

$$uF_{21}(\phi_1, \phi_2) + v\{F_{22}(\phi_1, \phi_2) - 4\pi^2\nu^2M\} = 0 \tag{2.6b}$$

where

$$F_{11}(\phi_1, \phi_2) = \sum_{\lambda,\mu} a_{\lambda,\mu}e^{i(\phi_1\lambda+\phi_2\mu)}; \qquad F_{12}(\phi_1, \phi_2) = \sum_{\lambda,\mu} b_{\lambda,\mu}e^{i(\phi_1\lambda+\phi_2\mu)} \tag{2.6c}$$

$$F_{21}(\phi_1, \phi_2) = \sum_{\lambda,\mu} a'_{\lambda,\mu}e^{i(\phi_1\lambda+\phi_2\mu)}; \qquad F_{22}(\phi_1, \phi_2) = \sum_{\lambda,\mu} b'_{\lambda,\mu}e^{i(\phi_1\lambda+\phi_2\mu)}. \tag{2.6d}$$

In order for solutions of (2.6) to exist, the determinant of the coefficients of u and v must vanish:

$$\begin{vmatrix} F_{11}(\phi_1, \phi_2) - 4\pi^2\nu^2M & F_{12}(\phi_1, \phi_2) \\ F_{21}(\phi_1, \phi_2) & F_{22}(\phi_1, \phi_2) - 4\pi^2\nu^2M \end{vmatrix} = 0. \tag{2.7}$$

This equation relates the frequency ν to the constants (ϕ_1, ϕ_2). There are two values of ν^2 for each (ϕ_1, ϕ_2) pair. Hence there is a total of $2N^2$ normal modes, or solutions (2.4) of our equations of motion (2.3). The most general solution is a linear combination of these $2N^2$ normal modes.

3. Curves of constant frequency in (ϕ_1, ϕ_2)-space and distribution of normal mode frequencies. Equation (2.7) is a quadratic equation in the square of the circular frequency $\omega = 2\pi\nu$:

$$\omega^4 - \omega^2(F_{11} + F_{22}) + (F_{11}F_{22} - F_{12}F_{21}) = 0. \tag{3.1}$$

The solutions of this equation are

$$[\omega(\phi_1, \phi_2)]^2 = \tfrac{1}{2}(F_{11} + F_{22}) \pm \tfrac{1}{2}[(F_{11} - F_{22})^2 + 4F_{12}F_{21}]^{1/2}. \tag{3.2}$$

Since each F_{ij} is a doubly periodic function of (ϕ_1, ϕ_2), ω also has this character and $\omega(\phi_1, \phi_2) = \omega(\phi_1 + 2\pi m, \phi_2 + 2\pi n)$ when m and n are positive or negative integers. We shall refer to all those frequencies which are generated as (ϕ_1, ϕ_2) runs through the N^2 values defined by (2.5) as a branch of the frequency spectrum. Our 2D lattice has two branches. We shall refer to the (ϕ_1, ϕ_2) space over which our doubly periodic function $\omega(\phi_1, \phi_2)$ is defined as a doubly periodic space.

Henceforth all of our discussions will be concerned with lattices which contain a very large number, N^2, of particles. Our various formulae will refer to limit results as $N \to \infty$. In this limit the uniformly distributed points in (ϕ_1, ϕ_2)-space (those defined by 2.5) which correspond to a branch of frequencies of normal modes become dense. The number of normal modes associated with a closed region in (ϕ_1, ϕ_2)-space is proportional to the area of that region. The proportionality constant is $N^2/4\pi^2$ since the area of a single period of our doubly periodic (ϕ_1, ϕ_2)-space is associated with N^2 normal modes.

It is clear that a doubly periodic function defined on a two dimensional space is equivalent to a function defined on a torus. Since the F_{ij}'s are continuous functions of (ϕ_1, ϕ_2), two neighboring points in (ϕ_1, ϕ_2)-space yield frequencies which differ only slightly from each other. Curves of constant frequency can be constructed in (ϕ_1, ϕ_2)-space. They are closed curves on the (ϕ_1, ϕ_2) torus. The curve $\omega = \omega_1 = \text{constant}$ separates the torus into two regions, one which corresponds to frequencies $\nu = \omega/2\pi$ which are less than $\nu_1 = \omega_1/2\pi$ and the other to frequencies greater than ν_1 (there may actually be several closed curves of a single frequency of interest; in this case the regions of frequencies $< \nu_1$ may not be connected).

If we let $N_\alpha(\nu)$ be the number of frequencies less than ν in the α-th branch, we have in the limit as $N \to \infty$

$$N_\alpha(\nu) = \frac{N^2}{4\pi^2} \int_R \int d\phi_1 d\phi_2 \tag{3.3}$$

where the integration extends over the set R of all values of (ϕ_1, ϕ_2) for which $\nu_\alpha(\phi_1, \phi_2)$ (the definition of $\nu_\alpha^2(\phi_1, \phi_2)$ is $\omega_\alpha^2(\phi_1, \phi_2)/2\pi$ with ω_α^2 given by (3.2)) is less than ν^2. Since the number of frequencies between ν and $\nu + d\nu$ in the α-th branch is

$$g_\alpha(\nu)d\nu = N_\alpha(\nu + d\nu) - N_\alpha(\nu) = (\partial N_\alpha/\partial\nu)d\nu,$$

we have

$$g_a(\nu) = \frac{\partial N_a}{\partial \nu} = \frac{N^2}{4\pi^2}\frac{\partial}{\partial \nu}\int_R\int d\phi_1 d\phi_2. \tag{3.4}$$

An alternative formula, which has been derived by Bowers and Rosenstock [6], in terms of δ-functions, allows one to express $g_a(\nu)$ as an integral over a complete period of (ϕ_1, ϕ_2)-space. Now we write

$$\begin{aligned} g_a(\nu)d\nu &= N_a(\nu + \tfrac{1}{2}d\nu) - N_a(\nu - \tfrac{1}{2}d\nu) \\ &= \frac{N^2}{4\pi^2}\int_S\int d\phi_1 d\phi_2, \end{aligned}$$

where S is the set of values of (ϕ_1, ϕ_2) for which

$$-\nu d\nu < \nu_a^2(\phi_1, \phi_2) - \nu^2 < \nu d\nu.$$

If $F_a(x)$ is a function with the property

$$F_a(x) = \begin{cases} 1 & \text{if} \quad -a < x < a \\ 0 & \text{if} \quad |x| > a \end{cases}$$

then

$$\int_{-\infty}^{\infty} \frac{1}{2a} F_a(x)dx = 1,$$

so that $(1/2a)F_a(x)$ approaches the delta function $\delta(x)$ as $a \to 0$. Since

$$g_a(\nu)d\nu = \frac{N^2}{4\pi^2}\int_{-\pi}^{\pi}\int F_{\nu+d\nu}[\nu_a^2(\phi_1, \phi_2) - \nu^2]d\phi_1 d\phi_2,$$

we have

$$g_a(\nu) = \frac{2\nu N^2}{4\pi^2}\int_{-\pi}^{\pi}\int \delta[\nu_a^2(\phi_1, \phi_2) - \nu^2]d\phi_1 d\phi_2. \tag{3.5}$$

The Fourier integral representation of the δ-function

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixy}dy$$

will sometimes be useful in the detailed calculation of (3.5). It leads to

$$g_a(\nu) = \frac{\nu N^2}{(4\pi)^2\pi}\int_{-\infty}^{\infty}\int_{-\pi}^{\pi}\int \exp\{iy[\nu_a^2(\phi_1, \phi_2) - \nu^2]\}dyd\phi_1 d\phi_2. \tag{3.6}$$

Obvious 3D generalizations of (3.4) and (3.6) exist.

4. The frequency spectrum of some special examples of two-dimensional lattices. We shall now apply the ideas developed in the two previous sections to the calculation of the frequency spectrum of 2D lattices whose vibrations are transverse (normal to the plane of the lattice) and of several special cases with vibrations in the plane.

A considerable insight into the behavior of curves of constant frequency and the FS can be achieved by studying a simple model in which each particle on a 2D square $N \times N$ lattice has only one degree of freedom. This we shall regard as the displacement of a particle in a direction normal to the equilibrium plane of the lattice. We shall postulate each particle to interact only with its nearest neighbors but in an unsymmetrical manner with the force constant for interactions along a row differing from that for interactions along a column.

We let $z_{l,m}$ be the displacement of the (l,m)-th particle from equilibrium, and choose the potential energy of interaction between particles to be

$$\Phi = \tfrac{1}{2}\alpha_1 \sum_{l,m} (z_{l,m} - z_{l,m+1})^2 + \tfrac{1}{2}\alpha_2 \sum_{l,m} (z_{l,m} - z_{l+1,m})^2. \tag{4.1}$$

The equations of motion of the particles on our lattice are

$$M \frac{d^2 z_{l,m}}{dt^2} = -\frac{\partial \Phi}{\partial z_{l,m}} = -2(\alpha_1 + \alpha_2) z_{l,m} + \alpha_1 (z_{l,m+1} + z_{l,m-1}) + \alpha_2 (z_{l+1,m} + z_{l-1,m}).$$

As usual we use the periodic boundary condition

$$z_{l,m} = z_{l+N,m} = z_{l,m+N} = z_{l+N,m+N}$$

and find the frequencies of normal modes to be given by

$$4\pi^2 \nu^2 M = 2(\alpha_1 + \alpha_2) - 2\alpha_1 \cos \phi_1 - 2\alpha_2 \cos \phi_2 \tag{4.2}$$

where

$$\phi_1 = 2\pi j/N, \quad \phi_2 = 2\pi k/N, \quad \text{and} \quad j, k = 0, 1, 2, \cdots, N-1. \tag{4.3}$$

If N is even we can choose j and k to run from $-\frac{1}{2}N+1$ to $\frac{1}{2}N$.

The largest frequency, ν_L, appears at $\phi_1 = \phi_2 = \pi$. Then

$$4\pi^2 M \nu_L^2 = 4(\alpha_1 + \alpha_2). \tag{4.4a}$$

Hence, if we let

$$r = \alpha_1/\alpha_2 \leqq 1 \quad \text{and} \quad f = \nu/\nu_L \tag{4.4b}$$

(4.2) becomes

$$(1 - 2f^2)(1 + r) = r \cos \phi_1 + \cos \phi_2. \tag{4.5}$$

Schematic curves of constant frequency are plotted in Figure 3.

We can apply equation (3.4) to the calculation of $g(\nu)$. A special integral must be evaluated in the three cases

i. $$f^2 \leqq r/(1+r)$$

ii. $$r/(1+r) \leqq f^2 \leqq 1/(1+r)$$

iii. $$1/(1+r) \leqq f^2 \leqq 1.$$

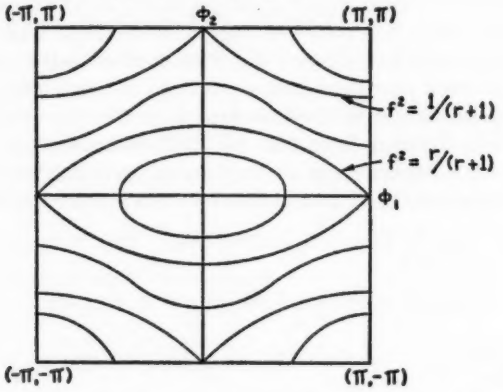


FIG. 3. Schematic curves of constant frequency in the reciprocal lattice of a two dimensional crystal.

The typical curves in one quadrant of each of three regimes are given in Figure 4. We now calculate the integrals of each of the shaded areas in the figures.

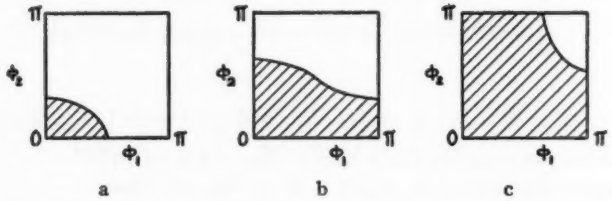


FIG. 4. Typical curves of constant frequency for three ranges of frequency.

The FS is proportional to the derivatives of these areas with respect to frequency (equation 3.4).

Case i. The intercept at $\phi_1 = 0$ is

$$\phi_2^{(0)} = \cos^{-1}\left[1 - 2f^2(1+r)\right]$$

while that at $\phi_2 = 0$ is

$$\phi_1^{(0)} = \cos^{-1}\left[1 - 2(1+r)f^2/r\right]$$

Hence the shaded area in Figure 4a is

$$\int_0^{\phi_1(0)} \cos^{-1}\{(1-2f^2)(1+r) - r\cos\phi_1\}d\phi_1$$

so that (remembering that each quadrant yields only one fourth of the frequencies)

$$\nu_{LG}(\nu) = \frac{N^2}{\pi^2}\frac{\partial}{\partial f}\int_0^{\phi_1(0)} \cos^{-1}\{(1-2f^2)(1+r) - r\cos\phi_1\}d\phi_1 = \frac{4N^2(1+r)f}{\pi^2} \times$$

$$\int_{r-2(1+r)f^2}^{r} \frac{dx}{\{(r-x)(r+x)[2(1+r)f^2 - r + x][2+r-2f^2(1+r)-x]\}^{1/2}}.$$

In our interval of interest $0 \leqq f^2 \leqq r/(1+r)$, the roots of the polynomial of the denominator of the integrand are ordered with

$$2 + r - 2f^2(1+r) > r \geqq r - 2f^2(1+r) \geqq -r.$$

Hence a direct application of equation 553 in Peirce's Table of Integrals yields

$$(4.6) \qquad \nu_{LG}(\nu) = \frac{4N^2(1+r)f}{\pi^2\sqrt{r}} K\left((1+r)f\sqrt{\frac{1-f^2}{r}}\right) \quad \text{if} \quad 0 \leqq f^2 \leqq r/(1+r)$$

where $K(k)$ is the complete elliptic integral

$$K(k) = \int_0^1 \frac{dx}{[(1-x^2)(1-k^2x^2)]^{1/2}}.$$

Case ii. Here we compute the shaded area in Figure 4b. This is

$$\int_0^r \cos^{-1}\{(1-2f^2)(1+r) - r\cos\phi\}d\phi_1.$$

Hence, if we let $x = r\cos\phi$, and apply (3.4) we find

$$(4.7) \qquad \nu_{LG}(\nu) = \frac{4N^2}{\pi^2(1-f^2)^{1/2}} K\left(\frac{r^{1/2}}{(1+r)f\sqrt{1-f^2}}\right)$$

$$\text{if} \quad r/(1+r) \leqq f^2 \leqq 1/(1+r).$$

Case iii. In the range $f^2 > 1/(1+r)$, the curves of constant frequency intersect the line $\phi_2 = \pi$ at

$$\phi_1^{(\pi)} = \cos^{-1}\{[(2+r) - 2f^2(1+r)]/r\}.$$

Hence the area of the shaded portion of Figure 4c is

$$\pi\phi_1^{(r)} + \int_{\phi_1^{(r)}}^{\pi} \cos^{-1}\{(1-2f^2)(1+r) - r\cos\phi_1\}\,d\phi_1$$

so that

$$\nu_L g(\nu) = \frac{4N^2(1+r)f}{\pi^2 r^{1/2}} K((1+r)f\sqrt{(1-f^2)/r}) \quad \text{if} \quad 1 \geqq f^2 \geqq 1/(1+r). \tag{4.8}$$

It is to be noted that (4.8) and (4.6) have exactly the same form.

In a symmetrical two-dimensional lattice $r=1$ and the range in which (4.7) exists vanishes. The elliptic integral $K(k)$ behaves as $\log [4/(1-k^2)^{1/2}]$ as $k\to 1$. This logarithmic singularity occurs in $g(\nu)$ as ν/ν_L approaches $[r/(1+r)]^{1/2}$ or $[1/(1+r)]^{1/2}$. In fact,

$$\nu_L g(\nu) \sim \begin{cases} -(N^2/\pi^2)(1+r)^{1/2}\log\{\nu/\nu_L - [r/(1+r)]^{1/2}\}^2 \\ \qquad\qquad \text{as } (\nu/\nu_L)^2 \to r/(1+r) & \text{(4.9a)} \\ -(N^2/\pi^2)r^{-1/2}(1+r)^{1/2}\log\{\nu/\nu_L - (1+r)^{-1/2}\}^2 \\ \qquad\qquad \text{as } (\nu/\nu_L)^2 \to 1/(1+r). & \text{(4.9b)} \end{cases}$$

As the lattice becomes more and more symmetrical, (*i.e.*, as $r\to 1$) the two logarithmic singularities approach each other and become one in the limit. In other words the singularity is split by reducing the symmetry. We have plotted $g(\nu)$ for several values of r in Figure 5 to exhibit this property.

The result for a symmetrical lattice agrees with that of Bowers and Rosenstock who showed that the logarithmic singularity persists even when next nearest neighbors are taken into account. As long as the nearest neighbor interactions predominate, the singularity moves toward higher frequencies with increasing next nearest neighbor force constants. Curves based on the formulae of Bowers and Rosenstock are given in Figure 6.

An especially important feature of the results given above (as well as of those of Bowers and Rosenstock [6]) is that the logarithmic singularities in the FS occur at frequencies which correspond to saddle points in (ϕ_1, ϕ_2)-space. In Figure 3 the saddle point on the $f^2=(1+r)^{-1}$ curve occurs at $(0, \pi)$ and that on the $f^2=r/(1+r)$ curve appears at $(\pi, 0)$. In a symmetrical lattice with $r=1$, both of these saddle points are on the curve with $f^2=\frac{1}{2}$.

Since $K(k)\sim\frac{1}{2}\pi$ as $k\to 0$, we have as $\nu\to 0$

$$\nu_L g(\nu) \sim 2N^2(1+r)\nu/\pi r^{1/2}\nu_L.$$

In the Debye continuum theory of a two dimensional lattice $g(\nu)$ is proportional to ν for all frequencies. It is not surprising that the continuum and discrete models give similar results in the range of wave lengths long compared to lattice spacings.

Equations (4.6)–(4.8) reduce properly to the one dimensional FS as $r\to 0$.

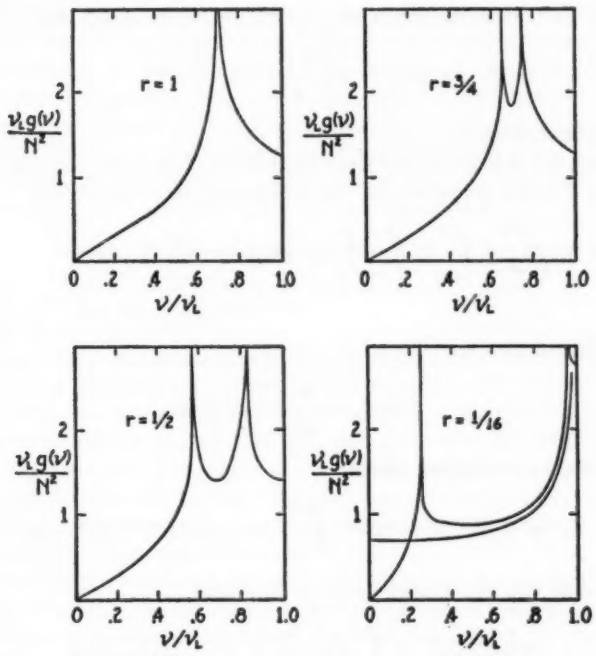


FIG. 5. Variation of frequency spectrum of two dimensional lattice with one degree of freedom per lattice point with increasing anisotropy. The parameter r equals 1 in an isotropic lattice and zero in the absence of coupling in one direction.

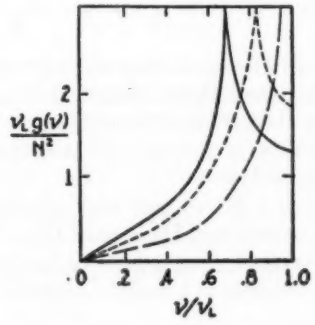


FIG. 6. Variation of frequency spectrum with ratio of next nearest to nearest neighbor force constants in two dimensional lattice with one degree of freedom per lattice point. Solid line to left represents no next nearest neighbor coupling. As this coupling increases the peak moves to the right. The curve whose peak falls at $\nu/\nu_L = 1$ corresponds to equal nearest and next nearest neighbor coupling.

In this limit (4.7) describes the entire spectrum. Since $K(0) = \frac{1}{2}\pi$, (4.7) becomes

$$\nu_L g(\nu) = N\left[\frac{2N}{\pi}(1 - f^2)^{-1/2}\right],$$

the one-dimensional $g(\nu)$.

Equations 4.6–4.8 could also have been derived from the formula (3.6) which involves the Fourier transform of the δ-function. In our problem

$$g(\nu) = \frac{N^2\nu}{\pi(4\pi)^2}\int_{-\infty}^{\infty} dy \int_{-\pi}^{\pi}\int \exp\{iy[\nu^2 - \nu_L^2\{(1 + r) - r\cos\phi_1 - \cos\phi_2\}/2(1 + r)]\}d\phi_1 d\phi_2. \tag{4.10}$$

An application of the formula

$$J_0(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\cos\theta}d\theta \tag{4.11}$$

and a simple transformation yields

$$\nu_L g(\nu) = 2N^2(1 + r)f\pi^{-1}\int_{-\infty}^{\infty} J_0(x)J_0(rx)e^{i(1+r)(2f^2-1)x}dx. \tag{4.12}$$

With the aid of the two Bessel function formulae

$$\pi J_0(z)J_0(Z) = \int_0^{\pi} J_0(\sqrt{Z^2 + z^2 - 2Zz\cos\theta})d\theta$$

and

$$\int_0^{\infty} J_0(at)\cos bt\, dt = \begin{cases}(a^2 - b^2)^{-1} & \text{if } a > b\\ 0 & \text{if } a < b,\end{cases}$$

(4.12) can easily be shown to be equivalent to (4.6)–(4.8).

We shall now show that the saddle points and logarithmic singularities discussed above also exist in 2D lattices of particles which vibrate in the plane of the lattice. It was in this type of system that the existence of the logarithmic singularities was first noticed [5].

The simplest example of a 2D system with vibrations in the plane is one with interactions between nearest neighbors only. One would expect the coupling between horizontal and vertical components of lattice vibrations to be very weak. On that basis let us consider as a model a system with the potential energy function

$$\Phi = \tfrac{1}{2}\alpha_1\sum_{l,m}(u_{l,m} - u_{l+1,m})^2 + \tfrac{1}{2}\alpha_2\sum_{l,m}(u_{l,m} - u_{l,m+1})^2 + \tfrac{1}{2}\beta_1\sum_{l,m}(v_{l,m} - v_{l,m+1})^2 + \tfrac{1}{2}\beta_2\sum_{l,m}(v_{l,m} - v_{l+1,m})^2. \tag{4.13}$$

The constants α_1 and β_1 are force constants associated with displacements parallel to the line connecting two particles while α_2 and β_2 go with displacements perpendicular to the connecting line. If all forces between particles are central, $\alpha_2=\beta_2=0$. Since the non-central contribution to forces in a crystal is usually small, we assume that $\alpha_2 \ll \alpha_1$ and $\beta_2 \ll \beta_1$. The 3D analogue of this model has been discussed by Newell and Rosenstock [13].

The characteristic determinant (2.7) for this system is diagonal. The reader can easily verify that the frequencies of normal modes are

$$4\pi^2\nu^2M = 2(\alpha_1 + \alpha_2) - 2\alpha_1 \cos \phi_1 - 2\alpha_2 \cos \phi_2$$
$$4\pi^2\nu^2M = 2(\beta_1 + \beta_2) - 2\beta_1 \cos \phi_2 - 2\beta_2 \cos \phi_1$$

where ϕ_1 and ϕ_2 satisfy (2.5).

Since these frequencies are generated by exactly the same equation, (4.2), as the generating equation for normal mode frequencies in a system with transverse vibrations, the FS, $g(\nu)$, must have the form as that plotted in Figure 5. In a symmetrical lattice $(\alpha_1, \alpha_2)=(\beta_1, \beta_2)$, and there are two logarithmic singularities as long as $\alpha_2 \neq \alpha_1$ (the case $\alpha_1=\alpha_2$ is quite unrealistic physically). In an unsymmetrical lattice with $(\alpha_1, \alpha_2) \neq (\beta_1, \beta_2)$, $\alpha_2 \neq \alpha_1$, $\beta_2 \neq \beta_1$, and none of the α's or β's equal to zero, there are four singularities; the removal of a symmetry doubles the number of singularities.

When the forces are central $(\alpha_2=\beta_2=0)$, $g(\nu)$ becomes exactly that of a one-dimensional system. In order to restore the two-dimensional character to a lattice of particles with central forces and vibrations in the lattice plane only, one must take into account interactions between more distant neighbors.

The calculation of $g(\nu)$ becomes rather complicated when interactions between more than nearest neighbors are included. The case of interactions between nearest and next nearest neighbors has been discussed at considerable length by the author [5]. Here we shall merely give a statement of the problem and summarize the results.

We restrict ourselves to symmetrical systems with central forces (with interactions only along lines which connect particles). A typical set of interactions are those along lines through the vectors given in Figure 7. The total potential energy of interaction between all particles is ($u_{l,m}$ and $v_{l,m}$ represent horizontal and vertical displacement of the (l, m)-th particle)

$$\begin{aligned}\Phi = {} & \tfrac{1}{2}\alpha \sum_{l,m} [(u_{l,m} - u_{l+1,m})^2 + (v_{l,m} - v_{l,m+1})^2] \\ & + \tfrac{1}{2}\gamma \sum_{l,m} [(u_{l,m} - u_{l+1,m+1} + v_{l,m} - v_{l+1,m+1})^2 \\ & \qquad + (u_{l,m} - u_{l+1,m-1} - v_{l,m} + v_{l+1,m-1})^2].\end{aligned} \tag{4.15}$$

In the usual manner the characteristic equation is found to be

$$\begin{vmatrix} (1-\tau)(1-c_1)+\tau(1-c_1c_2)-2f^2 & \tau s_1 s_2 \\ \tau s_1 s_2 & (1-\tau)(1-c_2)+\tau(1-c_1c_2)-2f^2 \end{vmatrix} = 0 \tag{4.16}$$

where

$$c_j = \cos \phi_j, \qquad s_j = \sin \phi_j, \quad \text{and} \quad f = \nu/\nu_L. \tag{4.17}$$

The largest frequency is given by

$$\nu_L = [(4\alpha + 8\beta)/4\pi^2 M]^{1/2} \tag{4.18}$$

and

$$\tau = 8\gamma/4\pi^2 M\nu_L^2 = 1/[1 + (\alpha/2\gamma)]. \tag{4.19}$$

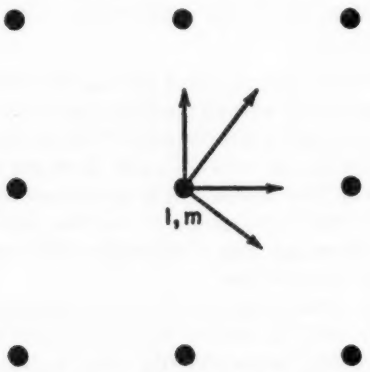


FIG. 7. Typical directions of interaction between nearest and next nearest neighbors in a two dimensional lattice.

The frequencies of normal modes are generated by

$$\begin{aligned} f^2 = (\nu/\nu_L)^2 &= \tfrac{1}{4}[2\tau(1 - c_1c_2) + (1 - \tau)(2 - c_1 - c_2)] \\ &\pm \tfrac{1}{4}[4\tau^2(c_2^2 - 1)(c_1^2 - 1) + (1 - \tau)^2(c_2 - c_1)^2]^{1/2} \end{aligned} \tag{4.20}$$

as the values of ϕ_j run through their usual range (2.5).

An unfortunate feature of (4.20) is that it involves radicals. The integration of (3.4) or (3.6) does not give a closed form result in terms of well known functions. However, when $\tau = 1/3$, the quantity inside the radical is a perfect square, and the two branches of frequencies become:

$$f_+^2 = \tfrac{1}{6}\{2(1 - c_1c_2) + 2 - c_1 - c_2\} \tag{4.21a}$$

$$f_-^2 = \tfrac{1}{6}\{2 - c_1 - c_2\}. \tag{4.21b}$$

The branch (4.21b) has a form that has already been discussed ($r = 1$ in (4.5)). The "+" branch can be analyzed easily and also leads to a FS that is a complete elliptic integral of the second kind. Both branches have logarithmic

singularities (the "+" branch at $f^2=3/4$ and the "−" branch at $f^2=1/3$) at values of f which correspond to saddle points. The curves of constant frequency (in one quadrant of (ϕ_1, ϕ_2)-space) of the "+" branch are drawn schematically in Figure 8.

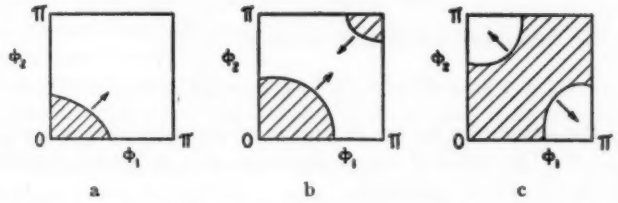


FIG. 8. Curves of constant frequency in the three ranges. The arrows point in the direction of increasing frequency.

Although one cannot obtain simple expressions for $g(\nu)$ for other values of τ, it is not difficult to plot the curves of constant frequency for any fixed τ. For all values of $\tau<1/2$ the "−" branch has a saddle point at $f^2=\tau$ and the "+" branch has one at $f^2=1-\tau$ if $\tau<1/5$ and at $f^2=(1+3\tau)^2/16\tau$ if $1/5<\tau<1/2$. It can be shown that logarithmic singularities exist at each of these saddle points and therefore that there is a singularity for each branch of the spectrum.

Smollett [7] has plotted out the curves of constant frequency for a lattice of alternate positively and negatively charged particles, all of equal mass, to determine the effect of long range coulomb forces on their character. These curves are somewhat more complicated than those with short range forces, but the saddle points and singularities still exist.

The qualitative similarity between the frequency spectra of all the models discussed leads one to ask three questions; (a) do saddle points generally lead to logarithmic singularities; (b) is there a physical reason for the existence of the saddle points; and (c) would the singularities remain if the periodic boundary conditions were replaced by some other ones? We shall now deduce an affirmative answer to all of these questions.

5. Topological basis for logarithmic singularities of the 2D FS.* We first show that saddle points in (ϕ_1, ϕ_2)-space always lead to logarithmic singularities in $g(\nu)$ (see also Smollett). If the saddle point is not on the ϕ_1 or ϕ_2 axis, let us rotate the coordinate system so that this becomes the case. If it is not symmetrical so that

$$\frac{\partial^2 f^2}{\partial \phi_1^2} = -\frac{\partial^2 f^2}{\partial \phi_2^2} = -2a^2 < 0, \tag{5.1}$$

* The topological discussion is too oversimplified for many applications to FS calculations. The reader is referred to the work of van Hove for a more detailed and rigorous analysis. The author is indebted to Professor van Hove for an informative discussion on which part of this section is based.

let the scale of one of the variables be changed so that this becomes true at the saddle point, $(\phi_1^{(s)}, 0)$. Since

$$\frac{\partial f^2}{\partial \phi_1} = \frac{\partial f^2}{\partial \phi_2} = \frac{\partial^2 f^2}{\partial \phi_1 \partial \phi_2} = 0$$

at $(\phi_1^{(s)}, 0)$, the Taylor expansion of f^2 about $(\phi_1^{(s)}, 0)$ is

$$f^2 = f_s^2 - a^2\left((\phi_1^{(s)} - \phi_1)^2 - \phi_2^2\right) + \cdots \tag{5.2}$$

where f_s is the value of ν/ν_L on the curve of constant frequency which goes through $(\phi_1^{(s)}, 0)$. The value of the ϕ_1 intercept, $\phi_1^{(0)}$, in the neighborhood of the saddle point is related to f through

$$\phi_1^{(s)} - \phi_1^{(0)} = (f_s^2 - f^2)^{1/2}/a. \tag{5.3}$$

The contribution of the shaded area of Figure 9 to $\nu_L g(\nu)$ in a range

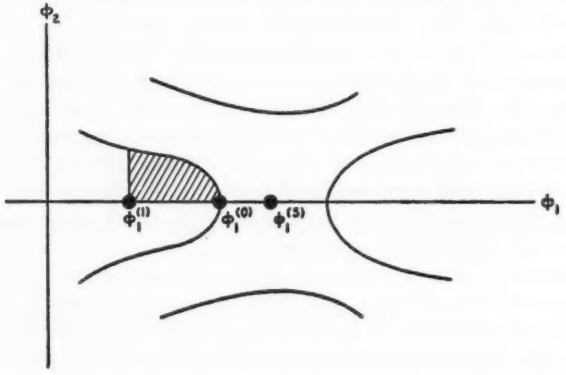


FIG. 9. Curves of constant frequency in the neighborhood of a saddle point.

$\phi_1^{(1)} \leqq \phi_1 \leqq \phi_1^{(0)}$ in which (5.2) is a good approximation to f^2 is

$$\begin{aligned}
&\frac{N^2}{4\pi^2}\frac{\partial}{\partial f}\int_{\phi_1^{(1)}}^{\phi_1^{(0)}} \left[(\phi_1^{(s)} - \phi_1)^2 - (f_2^2 - f^2)/a^2\right]^{1/2} d\phi_1 \\
&= \frac{N^2 f}{4\pi^2 a^2}\int_{\phi_1^{(1)}}^{\phi_1^{(0)}} \frac{d\phi_1}{\left[(\phi_1^{(s)} - \phi_1)^2 - (f_s^2 - f^2)/a^2\right]^{1/2}} \\
&= -\frac{N^2 f}{4\pi^2 a^2}\int_{\phi_1^{(s)} - \phi_1^{(1)}}^{(f_s^2 - f^2)^{1/2}/a} \frac{du}{\left[u^2 - (f_s^2 - f^2)/a^2\right]^{1/2}} \\
&= -\frac{N^2 f}{8\pi^2 a^2}\log(f_s - f) + \text{bounded function of } f.
\end{aligned} \tag{5.4}$$

As the frequency ν approaches ν_s (*i.e.*, as $f \to f_s$) this term contributes the logarithmic singularity to the FS as we set out to prove. In a similar manner it is easy to obtain the same result as ν_s is approached from higher frequencies.

Van Hove [10] has recently used the critical point theory of the calculus of variations in the large (developed by M. Morse [11]) to show that our saddle points are a direct consequence of the periodicity of a crystal lattice and the Born-Karman boundary conditions.

In preparation for the application of Van Hove's arguments we present a heuristic discussion of a special case of one of M. Morse's theorems. Let $G(\phi_1, \phi_2)$ be a continuous doubly periodic function of ϕ_1 and ϕ_2 with continuous first and second derivatives, the derivatives vanishing only at isolated points. Then Morse's theorem states that G has at least two saddle points. As was mentioned in section 3, a doubly periodic function is equivalent to a function defined on a torus; hence this theorem is equivalent to the statement that a function $G(\phi_1, \phi_2)$ with the properties postulated above and defined on a torus has at least two saddle points.

Since G has been chosen to be continuous it must have at least one maximum and one minimum point in each of its periods. Let the location of one of these maxima be denoted by an $\times$ in each period in Figure 10 and let the location of one set of equivalent minima be represented by heavy dots (points D, E, *etc.*)

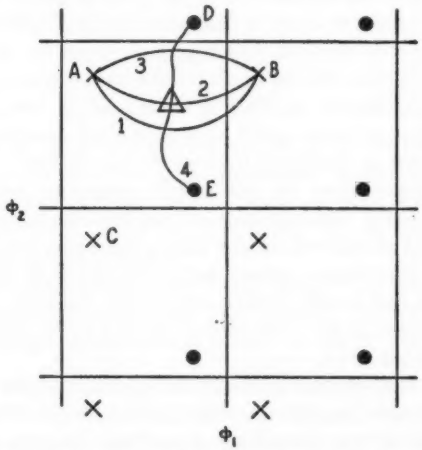


FIG. 10. Curves connecting maxima and minima of functions defined on a doubly periodic space.

If the maxima A and B are connected by a curve such as 1 in Figure 10, there is at least one point on the curve where G has a smaller value than at neighboring points on the curve. G also achieves a smallest value along any other curves, for example on 2 and 3, which connect A and B. The locus of these points forms a continuous curve which passes through E and D. G must achieve

a largest value somewhere along this locus of smallest values. Let this be accomplished at the point represented by the small triangle in Figure 10. This point must be a saddle point because as one passes along 4 from E to D it is a relative maximum, and as one passes from A to B it is a relative minimum.

The same argument could be applied by the examination of paths which connect A and C. Hence the function G has at least two saddle points. It is easy to trace out the same proof when one period of (ϕ_1, ϕ_2)-space is wrapped into a torus. Those paths which start at × and loop the torus, as in Figure 11a, correspond to paths from A to C in Figure 10, while those that go around the

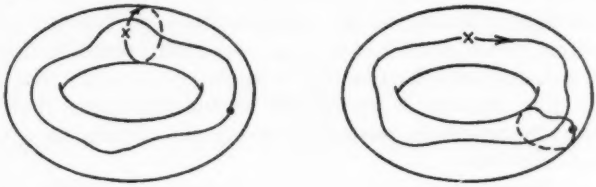


FIG. 11. Closed curves through maxima and minima of a function defined on a torus.

hole (as in Figure 11b) correspond to paths from A to B. No matter how the path which goes around the hole is deformed or displaced it cannot be transformed continuously into the one which goes through the hole. The number of saddle points of a function defined at all points on the surface of a 3D body is at least equal to the maximum number of closed paths which cannot be transformed into each other or into a point by a continuous deformation of the paths. This number is known as the Betti number of the surface.

The fact that there might be two saddle points in our reciprocal (ϕ_1, ϕ_2)-space does not guarantee the existence of two logarithmic singularities in $g(\nu)$. The value of $\nu(\phi_1, \phi_2)$ at the two saddle points may correspond to the same frequency. This is the case when a square lattice is elastically symmetrical. We see in Figure 3 that the two saddle points, one at $(\phi_1, \phi_2) = (0, \pi)$ and the other at $(\pi, 0)$ (as well as those at $(0, -\pi)$ and $(-\pi, 0)$) both correspond to the frequency $\nu = (\frac{1}{2})^{1/2}\nu_L$ when $r = 1$.

On the basis of the general theorem concerning saddle points, Van Hove has pointed out that the logarithmic singularities in the FS of a 2D lattice are a direct consequence of the postulate of the harmonicity of the interatomic forces and of the doubly periodic structure of the lattice. The number of interacting neighbors and the detailed nature of the force constants is not critical as long as the series F_{ij} in (3.6) converge and as long as the lattice is not so weakly coupled that the relaxation of an initial displacement of a given atom propagates a wave only along a single row and column (as is the case of a square lattice with central force interactions which act only between nearest neighbors).

The last of our questions about the singularities of the 2D $g(\nu)$ is whether their existence is sensitive to the choice of periodic boundary conditions. To show that the detailed choice of boundary conditions is unimportant we apply the following argument of Ledermann [15]. It is based on Ledermann's theorem, *If, in a Hermitian matrix the elements of r rows and their corresponding columns are modified in any way whatsoever, provided only that the matrix remains Hermitian, then the number of characteristic values which lie in any interval cannot increase or decrease by more than 2r.*

The particles in an $N\times N$ square lattice with interactions between nearest and next nearest neighbors can be divided into $(N-2)^2$ internal particles and $(4N-4)$ boundary particles. We might identify the internal ones by the numbers $1, 2, \cdots, (N-2)^2$ and the boundary ones by $(N-2)^2+1, \cdots, N^2$. The equations of motion of the internal particles are all of the same form.

The matrix of the secular determinant can then be expressed in the form

$$\left(\begin{array}{c|c} M_{ii} & M_{ib} \\ \hline M_{bi} & M_{bb} \end{array}\right) \begin{array}{l} 2(N-2)^2 \text{ rows.} \\ \hline 2(4N-4) \text{ rows.} \end{array} \tag{5.5}$$

where M_{ii} represents the matrix elements of the interactions between internal particles, M_{ib} and M_{bi} those between internal and boundary particles, and M_{bb} those between boundary particles. The corresponding matrix for Born-Karman boundary conditions would be of the form

$$\begin{pmatrix} M_{ii} & B_{ib} \\ B_{bi} & B_{bb} \end{pmatrix} \tag{5.6}$$

and hence differs from (5.5) in the elements of $2(4N-4)$ rows. Then the maximum number of normal modes which move into or out of a given frequency interval is at most $16(N-1)$ out of the order of $2N^2$. Hence in the limit as $N\to\infty$ a negligible fraction make the move, and in this limit $g(\nu)/2N^2$ is independent of the boundary conditions.

Even in systems with longer (but fixed) range interactions, the boundary thickness is independent of N so that the number of rows changed in (5.5) in going to periodic boundary conditions is still of $O(N)$.

6. The nature of the singularities in three-dimensional lattices. Van Hove has also discussed the singularities of the 3D FS by topological considerations. The reciprocal lattice of a 3D one component cubic crystal is also 3D. For convenience we shall restrict ourselves to simple cubic lattices. We let $\phi_3=2\pi l/N_3$, with l ranging from 1 to N_3, be the third dimension in the RL. Then each normal mode frequency is a triply periodic function of the ϕ's, $\nu(\phi_1, \phi_2, \phi_3)$. The 3D RL can be visualized as a set of parallel layers of 2D RL's with ϕ_3 constant in each layer. The complete FS of a crystal is the sum of the spectra of each layer in the RL.

The Morse theorem is applicable to each of these layers. The FS associated

with each has at least one or two saddle points, the minimum number depending on symmetry. If the saddle point remains at the same frequency on all layers, the total FS, being a sum of functions each with a logarithmic singularity at the same point, has a logarithmic singularity. This, however, is an exceptional case which occurs only when the force constants satisfy special conditions.

Generally the position of the saddle point varies from layer to layer. A schematic plot of $\nu_c(\phi_3)$, the singular frequency, is made in Figure 12. This frequency, being a continuous periodic function of ϕ_3 has at least one maximum (say $\nu_c^{(2)}$ at $\phi_3^{(2)}$) and one minimum (say $\nu_c^{(1)}$ at $\phi_3^{(1)}$). In the case of a single maxi-

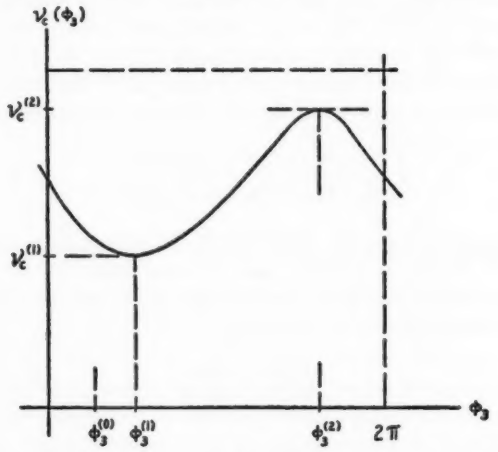


FIG. 12. Variation of singular frequency with layer.

mum and minimum all the logarithmic singularities are located between $\nu_c^{(1)}$ and $\nu_c^{(2)}$. Hence we can expect $g(\nu)$ to have a sharp rise in the neighborhood of $\nu_c^{(1)}$ and a sharp drop in the neighborhood of $\nu_c^{(2)}$. In the limit as $N_3 \to \infty$ these singularities become dense so that if one divides the FS by N_3 as the limit is taken, the final $g(\nu)$ should vary only slowly in the frequency interval $(\nu_c^{(1)}, \nu_c^{(2)})$ (Figs. 13 and 14).

We shall now investigate this behavior analytically for frequencies near $\nu_c^{(1)}$ and $\nu_c^{(2)}$. The singular frequencies on layers of the RL with ϕ_3 close to $\phi_3^{(1)}$ are given by

$$\nu_c(\phi_3) = \nu_c^{(1)} + b^2(\phi_3 - \phi_3^{(1)})^2 + \cdots \tag{6.1}$$

where

$$b^2 = \frac{1}{2} \frac{\partial^2 \nu_c(\phi_3)}{\partial \phi_3^2}\Bigg]_{\phi_3^{(1)}} > 0.$$

The FS of a ϕ_3 layer close to $\phi_3^{(1)}$ and at frequencies near $\nu_c^{(1)}$ is proportional to (see (5.4))

$$-\log|\nu_c(\phi_3)-\nu|. \tag{6.2}$$

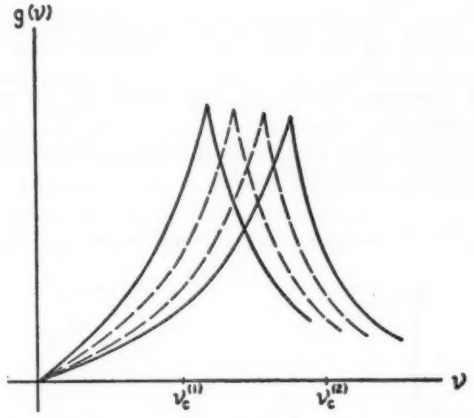


FIG. 13. Frequency spectra associated with various layers in a thin crystal.

We choose $\phi_3^{(0)}$ to be an arbitrary number such that (6.1) is valid for all ϕ_3 in the range $\phi_3^{(0)}<\phi_3<\phi_3^{(1)}$. Then the total contribution of the region of the RL in the neighborhood of the saddle points near $\phi_3^{(1)}$ to the frequency spectrum is proportional to

$$-2\frac{N_3}{2\pi}\int_{\phi_3^{(0)}}^{\phi_3^{(1)}}\log|\nu_c(\phi_3)-\nu|\,d\phi_3 \tag{6.3}$$

for those frequencies near $\nu_c^{(1)}$. The factor 2 is introduced so that the integration can stop at $\phi_3^{(1)}$ instead of proceeding to a distance $\phi_3^{(1)}-\phi_3^{(0)}$ beyond $\phi_3^{(1)}$.

We shall now show that the FS rises sharply as $\nu\to\nu_c^{(1)}$ (see Fig. 14) from the left. When $\nu<\nu_c^{(1)}$, (6.3) becomes

$$\begin{aligned} I_1 &= -\frac{N_3}{\pi}\int_{\phi_3^{(0)}}^{\phi_3^{(1)}}\log\left[(\nu_c^{(1)}-\nu)+b^2(\phi_3-\phi_3^{(1)})^2\right]d\phi_3 \\ &= -\frac{N_3}{\pi b}\int_0^c\log(a^2+u^2)\,du \\ &= -\frac{N_3}{\pi b}\left\{c\log(a^2+c^2)-2c+2a\tan^{-1}c/a\right\} \end{aligned}$$

where

$$a = |\nu_c^{(1)} - \nu|^{\frac{1}{2}},\ u = b(\varphi_3^{(1)} - \varphi_3),\ \text{and}\ c = (\phi_3^{(1)} - \phi_3^{(0)})b > 0.$$

In the limit as $\nu \to \nu_c^{(1)}$ (since $\tan^{-1} \infty = \frac{1}{2}\pi$)

$$I_1 \sim \frac{N_3}{\pi b}\left\{-\pi(\nu_c^{(1)} - \nu)^{1/2} + 2c(1 - \log c)\right\}. \tag{6.4}$$

Since the slope of I_1 is proportional to $1/(\nu_c^{(1)} - \nu)^{1/2}$, the FS, $g(\nu)$, has a vertical tangent at $\nu = \nu_c^{(1)}$ (see Fig. 14).

It can now be shown that $g(\nu)$ is continuous at $\nu_c^{(1)}$ but that $g'(\nu)$ is discontinuous. When $\nu > \nu_c^{(1)}$ we define ϕ_3^* so that

$$\nu - \nu_c^{(1)} = b^2(\phi_3^{(1)} - \phi_3^*)^2 = a^2.$$

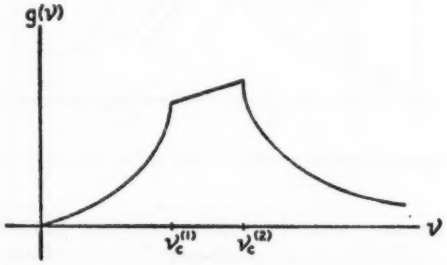


FIG. 14. Typical peak in a 3-D frequency spectrum.

Then (6.3) becomes

$$\begin{aligned} I_2 &= -\frac{N_3}{\pi}\left\{\int_{\phi_3^{(0)}}^{\phi_3^*} \log\left[b^2(\phi_3^{(1)} - \phi_3)^2 - a^2\right]d\phi_3 \right. \\ &\quad \left. + \int_{\phi_3^*}^{\phi_3^{(1)}} \log\left[a^2 - b^2(\phi_3^{(1)} - \phi_3)^2\right]d\phi_3\right\} \\ &= \frac{N_3}{\pi b}\left\{\int_c^a \log(u^2 - a^2)du + \int_a^0 \log(a^2 - u^2)du\right\} \\ &= -\frac{N_3}{\pi b}\left\{c\log(c^2 - a^2) + a\log\left[(c + a)/(c - a)\right] - 2c\right\} \\ &\sim \frac{N_3}{\pi b}\left\{2c(1 - \log c) - (\nu - \nu_c^{(1)})/c + O(\nu - \nu_c^{(1)})^{3/2}\right\}; \end{aligned} \tag{6.5}$$

as $\nu \to \nu_c^{(1)}$, (6.4) and (6.5) approach each other. Hence $g(\nu)$ is continuous at $\nu_0^{(1)}$. However, $\partial I_1/\partial\nu \to \infty$ when $\nu \to \nu_c^{(1)}$ from the left while $\partial I_2/\partial\nu \to -N_2/\pi bc$, a

finite number, as $\nu \rightarrow \nu_c^{(1)}$ from the right. Hence $g'(\nu)$ is discontinuous.

One can proceed with a similar analysis to find that $g(\nu)$ has a rapid drop with a vertical tangent at $\nu_c^{(2)}$, the location of the highest frequency saddle point in the RL. This behavior is exhibited in Figure 14.

The 3D RL can also be built up in layers which are normal to either the ϕ_1 or ϕ_2 axes. When the surfaces of constant frequency are not symmetrical functions of ϕ_1, ϕ_2 and ϕ_3 the argument used above can be repeated in each of the new alternative layer constructions to locate other singularities.

Let us consider a case which has symmetrical surfaces of constant frequency and which has simple peaks of the type given in Figure 14. Then let a symmetry be removed by making a small change in the force constants so that contours in planes normal to two of the axes are identical but different from those on planes perpendicular to the third axis. The total FS is the superposition of pairs of peaks with their edges slightly displaced from each other as in Figures

FIG. 15. Displacement of peaks by slight perturbation in symmetry.

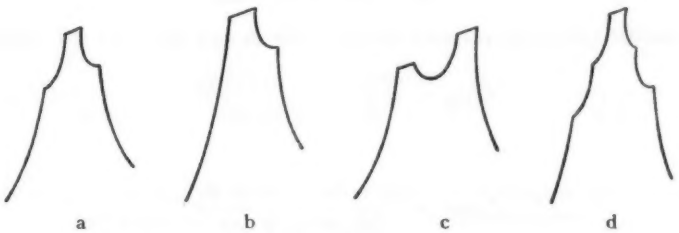


FIG. 16. Typical peaks in the FS for anisotropic 3-D crystals.

15a or 15b. The total curve, being the sum of these pairs, generally has peaks of the form of those in Figure 16a. Under special conditions only one of the edges might be displaced by the perturbation of the force constants, thus yielding curves such as those in Figure 16b. When larger perturbations are made one of the members of the pair of peaks may be displaced sufficiently to split the original peak into two as in Figure 16c.

Finally when small displacements in force constants are made which remove all symmetries with respect to ϕ_1, ϕ_2, and ϕ_3, three layered peaks such as that in Figure 16d may appear.

We shall now examine some models in which these effects can be studied analytically.

7. Some examples of 3D frequency spectra. As in the case of 2D lattices we shall first discuss a model in which one degree of freedom is associated with each lattice point in a $N\times N\times N$ 3D lattice with interactions between nearest neighbors. If the potential energy of interaction is chosen to be

$$\Phi = \tfrac{1}{2}\alpha_1 \sum_{l,m,n} (z_{l,m,n} - z_{l,m,n+1})^2 + \tfrac{1}{2}\alpha_2 \sum_{l,m,n} (z_{l,m,n} - z_{l,m+1,n})^2 + \tfrac{1}{2}\alpha_3 \sum_{l,m,n} (z_{l,m,n} - z_{l+1,m,n})^2 \tag{7.1}$$

and periodic boundary conditions are employed, the frequencies of normal modes are given by

$$4\pi^2\nu^2 M = 2(\alpha_1 + \alpha_2 + \alpha_3) - 2\alpha_1 \cos\phi_1 - 2\alpha_2 \cos\phi_2 - 2\alpha_3 \cos\phi_3 \tag{7.2}$$

where

$$\phi_1 = 2\pi j/N, \qquad \phi_2 = 2\pi k/N, \qquad \phi_3 = 2\pi s/N, \qquad j, k, s = 0, 1, 2, \cdots, N-1.$$

As usual we let $f = \nu/\nu_L$ so that

$$(1 - 2f^2)(\alpha_1 + \alpha_2 + \alpha_3) = \alpha_1 \cos\phi_1 + \alpha_2 \cos\phi_2 + \alpha_3 \cos\phi_3 \tag{7.2a}$$

with

$$4\pi^2 \nu_L^2 M = 4(\alpha_1 + \alpha_2 + \alpha_3). \tag{7.2b}$$

The saddle points on planes normal to the ϕ_3 axis occur at the frequencies

$$f_c^2(\phi_3) = \frac{\alpha_3 + 2\alpha_2 - \alpha_3 \cos\phi_3}{2(\alpha_1 + \alpha_2 + \alpha_3)} \tag{7.3}$$

and

$$f_c^2(\phi_3) = \frac{\alpha_3 + 2\alpha_1 - \alpha_3 \cos\phi_3}{2(\alpha_1 + \alpha_2 + \alpha_3)}. \tag{7.4}$$

The largest value of f_c^2 at a saddle point associated with (7.3) is

$$(f_c^{(1)})^2 = (\alpha_2 + \alpha_3)/(\alpha_1 + \alpha_2 + \alpha_3) \tag{7.5a}$$

while the smallest is

$$(f_c^{(1)})^2 = \alpha_2/(\alpha_1 + \alpha_2 + \alpha_3). \tag{7.5b}$$

The edges of the peak associated with (7.4) are at

$$(f_c^{(2)})^2 = (\alpha_1 + \alpha_3)/(\alpha_1 + \alpha_2 + \alpha_3) \tag{7.6a}$$

on the right and

$$(f_c^{(2)})^2 = \alpha_1/(\alpha_1 + \alpha_2 + \alpha_3) \tag{7.6b}$$

on the left. A similar analysis of saddle points on planes normal to the ϕ_1 and ϕ_2 axes yield the following four peaks with edges at f^2 equal to

$$\alpha_1/(\alpha_1 + \alpha_2 + \alpha_3) \quad \text{and} \quad (\alpha_1 + \alpha_2)/(\alpha_1 + \alpha_2 + \alpha_3) \tag{7.7}$$

$$\alpha_3/(\alpha_1 + \alpha_2 + \alpha_3) \quad \text{and} \quad (\alpha_3 + \alpha_2)/(\alpha_1 + \alpha_2 + \alpha_3) \tag{7.8}$$

$$\alpha_2/(\alpha_1 + \alpha_2 + \alpha_3) \quad \text{and} \quad (\alpha_2 + \alpha_1)/(\alpha_1 + \alpha_2 + \alpha_3) \tag{7.9}$$

$$\alpha_3/(\alpha_1 + \alpha_2 + \alpha_3) \quad \text{and} \quad (\alpha_3 + \alpha_1)/(\alpha_1 + \alpha_2 + \alpha_3). \tag{7.10}$$

The qualitative appearance of the FS depends on the particular values assigned to the force constants α_1, α_2, and α_3. We shall consider several interesting special cases.

When $\alpha_1=\alpha_2=\alpha_3$ we get a single peak with edges at $f^2=1/3$ and $f^2=2/3$ as in Figure 17a (and as was first obtained by Bowers and Rosenstock [6]).

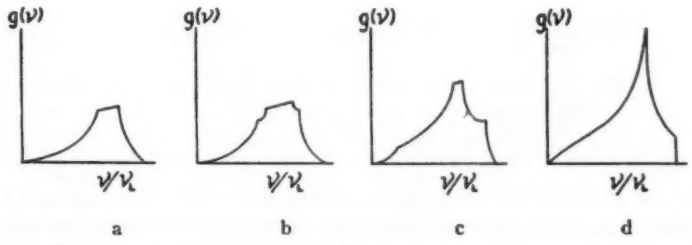


FIG. 17. Variation of frequency spectrum of 3-D lattice with one degree of freedom per lattice point with the reduction of coupling between layers.

If we let $\alpha_3=\alpha_2=\alpha$ and $\alpha_1=\alpha(1+\delta)$ with $|\delta| \ll 1$, the peak edges appear at f^2 equal to $(\frac{1}{3}, \frac{2}{3})$, $(\frac{1}{3}(1+\delta), \frac{1}{3}(2+\delta))$ $(\frac{1}{3}, \frac{1}{3}(2+\delta))$, so that the FS has the form of Figure 17b.

If $\delta=\epsilon-1$ with ϵ a small positive number the edges are at $(\frac{1}{3}, \frac{2}{3})$; $(\frac{1}{3}\epsilon, \frac{1}{3}(1+\epsilon))$; $(\frac{1}{3}, \frac{1}{3}(1+\epsilon))$. In the limit as $\epsilon \to 0$ the system degenerates into a set of independent layers while the peak of the FS narrows (Fig. 17c) until in the limit it has the 2D logarithmic singularity (17d).

The peak in 17a splits into two when $\delta>1$, for then $\frac{1}{3}(1+\delta)>\frac{2}{3}$ and the FS has the appearance of Figure 18a.

Let us now consider a model which is more representative of a real crystal; namely, a simple cubic lattice of springs and masses with three degrees of

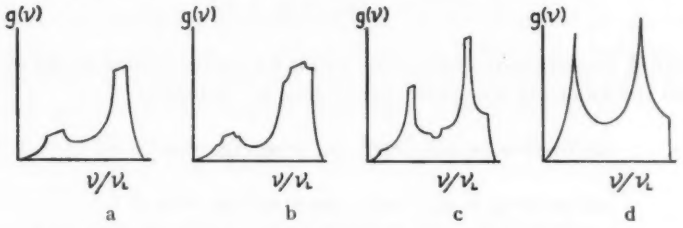


FIG. 18. Variations of frequency spectrum of simple cubic lattice (with interaction between nearest neighbors only) as coupling between layers weakens.

freedom each and with interactions between nearest neighbors only. Then generalizing (4.13) we choose the potential energy function to be

$$\begin{aligned}\Phi = {} & \tfrac{1}{2}\alpha_1 \sum_{l,m,n} (u_{l,m,n} - u_{l+1,m,n})^2 + \tfrac{1}{2}\alpha_2 \sum_{l,m,n} (u_{l,m,n} - u_{l,m+1,n})^2 \\ & + \tfrac{1}{2}\alpha_3 \sum_{l,m,n} (u_{l,m,n} - u_{l,m,n+1})^2 \\ & + \tfrac{1}{2}\beta_1 \sum_{l,m,n} (v_{l,m,n} - v_{l,m+1,n})^2 + \tfrac{1}{2}\beta_2 \sum_{l,m,n} (v_{l,m,n} - v_{l,m,n+1})^2 \\ & + \tfrac{1}{2}\beta_3 \sum_{l,m,n} (v_{l,m,n} - v_{l+1,m,n})^2 \\ & + \tfrac{1}{2}\gamma_1 \sum_{l,m,n} (w_{l,m,n} - w_{l,m,n+1})^2 + \tfrac{1}{2}\gamma_2 \sum_{l,m,n} (w_{l,m,n} - w_{l+1,m,n})^2 \\ & + \tfrac{1}{2}\gamma_3 \sum_{l,m,n} (w_{l,m,n} - w_{l,m+1,n})^2.\end{aligned}$$

The u, v, and w's represent components of displacements in the x, y, and z directions while α_1, β_1, and γ_1 are force constants associated with displacements in the direction of the lines which connect particles. If forces are central all other α, β, and γ's vanish. The introduction of periodic boundary conditions yields the following normal mode frequencies:

$$(7.11)\quad 4\pi^2\nu^2 M = 2(\alpha_1 + \alpha_2 + \alpha_3) - 2\alpha_1 \cos\phi_1 - 2\alpha_2 \cos\phi_2 - 2\alpha_3 \cos\phi_3$$

$$(7.12)\quad 4\pi^2\nu^2 M = 2(\beta_1 + \beta_2 + \beta_3) - 2\beta_3 \cos\phi_1 - 2\beta_1 \cos\phi_2 - 2\beta_2 \cos\phi_3$$

$$(7.13)\quad 4\pi^2\nu^2 M = 2(\gamma_1 + \gamma_2 + \gamma_3) - 2\gamma_2 \cos\phi_1 - 2\gamma_3 \cos\phi_2 - 2\gamma_1 \cos\phi_3;$$

the ϕ's have the usual range. These equations are of the same form as (7.2). Hence the discussion given above is immediately applicable.

First we consider the symmetrical simple cubic lattice with $\alpha_1=\beta_2=\gamma_3=\alpha(1+\delta)$ and $\alpha_2=\alpha_3=\beta_1=\beta_3=\gamma_1=\gamma_2=\alpha$. Since the noncentral character of the forces acting on atoms in a solid is small, we assume $\delta \gg 0$. When $\delta > 1$ two peaks exist as is shown in Figure 18a. Detailed calculations have been made on this

model by Newell and Rosenstock [13]. This type of FS is typical for single cubic lattices with interactions between more distant neighbors. Newell [12] has shown this through several exact numerical calculations.

It is interesting to see how the FS in Figure 18a degenerates into that of a two dimensional lattice as the interactions between layers are made weaker. We let $\alpha_1=\gamma_3=\alpha(1+\delta)$, $\beta_2=\epsilon\alpha(1+\delta)$, $\beta_1=\alpha_2=\beta_3=\gamma_2=\alpha$ and $\gamma_1=\alpha_3=\epsilon\alpha$ and let $\epsilon\to 0$. Then the peaks become progressively narrower until in the limit they become logarithmic.

If one wishes to find the FS of a real crystal, several courses are available. He should first locate the various singularities and the low frequency behavior by analytical methods. Then either by numerical integrations based on the 3D analogues of (3.4) or (3.6) or by actual calculation and sorting of normal mode frequencies on a high speed computing machine, a FS curve can be sketched which is consistent with the location of singularities. Another scheme is to choose an empirical function which is a linear combination of one with properly located singularities and a polynomial (or the product of these two types of functions) and find the values of the constants which lead to independently calculated moments of the FS. These moments are easily obtained from the characteristic equations which define the normal mode frequencies [16].

References

1. A. Einstein, Ann. Phys., vol. 22, 1907, p. 180.
2. P. Debye, Ann. Phys., vol. 39, 1912, p. 789.
3. M. Blackman, Reports on Progress in Physics, vol. 8, 1942, p. 11.
4. M. Born and von Karman, Phys. Zeit., vol. 13, 1912, p. 297.
5. E. Montroll, J. Chem. Phys., vol. 15, 1947, p. 575.
6. W. Bowers and H. Rosenstock, J. Chem. Phys., vol. 18, 1950, p. 1056.
7. M. Smollett, Proc. of Phys. Soc. (London) A65, 1952, p. 109.
8. J. Holsen and W. Nierenberg, Phys. Rev., vol. 89, 1953, p. 662.
9. H. B. Rosenstock, J. Chem. Pys., vol. 21, 1953, p. 2064.
10. L. van Hove, Phys. Rev., vol. 89, 1953, p. 1189.
11. M. Morse, Trans. Am. Math. Soc., vol. 27, 1925, p. 345; Calculus of Variations in the Large, Colloquium Lectures, Am. Math. Soc., 1934.
12. G. Newell, J. Chem. Phys., vol. 21, 1953, p. 1877.
13. H. B. Rosenstock and G. Newell, J. Chem. Phys., vol. 21, 1953, p. 1607.
14. H. B. Rosenstock and H. M. Rosenstock, J. Chem. Phys., vol. 21, 1953, p. 1608.
15. W. Ledermann, Proc. Roy. Soc., A 182, 1944, p. 362.
16. E. Montroll, J. Chem. Phys., vol. 10, 1942, p. 219; vol. 11, 1943, p. 481.

THE SLAUGHT PAPERS

The Herbert Ellsworth Slaught Memorial Papers are a series of brief expository pamphlets published as supplements to the American Mathematical Monthly. Two numbers have appeared:

1. *Fourier's Series; The Genesis and Evolution of a Theory* by R. E. Langer. v + 86 pages. Paper.

2. *Outline of the History of Mathematics (6th edition)* by R. C. Archibald. iv + 114 pages. Paper.

Copies at one dollar each postpaid may be ordered from:

The Mathematical Association of America
University of Buffalo
Buffalo 14, New York

Now available:

PROFESSIONAL OPPORTUNITIES IN MATHEMATICS

Second Edition: January 1954

A Publication of the Arnold Buffum Chace Fund

The report of a Committee of the Mathematical Association of America, consisting of H. W. Brinkmann, Z. I. Mosesson, S. A. Schelkunoff, S. S. Wilks, and Mina Rees, Chairman.

24 pages, paper covers

25¢ for single copies; 10¢ each for orders of ten or more

Send orders to: Harry M. Gehman, Secretary-Treasurer
Mathematical Association of America
University of Buffalo
Buffalo 14, New York

Index of Volumes 1 to 56 Inclusive

The American Mathematical Monthly

Copies at one dollar each postpaid may be ordered from:

H. M. Gehman, Secretary-Treasurer
Mathematical Association of America

University of Buffalo **Buffalo 14, New York**

THE AMERICAN MATHEMATICAL MONTHLY

●

Complete volumes from volume 23 (1916) to volume 52 (1945) inclusive, are for sale at the following rates:

Single volumes:	$10 per volume
Five or more volumes (any years):	$ 5 per volume

We pay transportation charges if payment accompanies order.

Send orders to:

Harry M. Gehman, Secretary-Treasurer
Mathematical Association of America
University of Buffalo
Buffalo 14, New York

THE CARUS MONOGRAPHS

•

The Carus Monographs are expository presentations of the best thought and keenest researches in pure and applied mathematics, set forth in a manner comprehensible not only to teachers and students specializing in mathematics, but also to scientific workers in other fields. They are intended especially for the wide circle of thoughtful people who, having a moderate acquaintance with elementary mathematics wish to extend their knowledge without prolonged and critical study of the mathematical journals and treatises.

•

The complete list of Carus Monographs is:

No. 1. Calculus of Variations, by G. A. Bliss. 1925.

No. 2. Analytic Functions of a Complex Variable, by D. R. Curtiss. 1926.

No. 3. Mathematical Statistics, by H. L. Rietz 1927.

No. 4. Projective Geometry, by J. W. Young. 1930.

No. 5. History of Mathematics in America before 1900, by D. E. Smith and Jekuthiel Ginsburg. 1934.

No. 6. Fourier Series and Orthogonal Polynomials, by Dunham Jackson. 1941.

No. 7. Vectors and Matrices, by C. C. MacDuffee. 1943.

No. 8. Rings and Ideals, by N. H. McCoy. 1948.

No. 9. The Theory of Algebraic Numbers, by Harry Pollard. 1950.

No. 10. The Arithmetic Theory of Quadratic Forms, by B. W. Jones. 1950.

The price of each monograph is $1.75 per copy to members of the Mathematical Association, one copy to each member, when ordered directly through the office of the Secretary of the Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

Additional copies and copies for non-members of Monographs 1-8 are priced at $3.00 each, and must be purchased from the Open Court Publishing Co., LaSalle, Illinois.

In the case of Monographs 9 and 10, additional copies and copies for nonmembers must be purchased at $3.00 from John Wiley and Sons, 440 Fourth Ave., New York 16, N.Y.